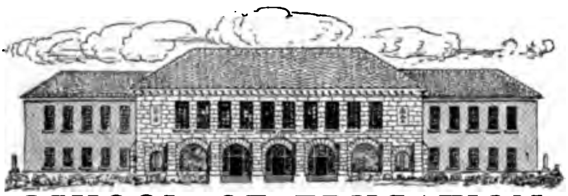

Ernest J. Page,
Stanford '00.

A TEXT-BOOK

OF

GENERAL PHYSICS

FOR THE USE OF COLLEGES AND SCIENTIFIC SCHOOLS

BY

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BOSTON, U.S.A.
GINN & COMPANY, PUBLISHERS
The Athenæum Press
1899

PREFACE.

THE most marked improvement in the method of recent text-books on physics is the growing tendency to emphasize the essential continuity of the science. It is true that physics, as a matter of convenience, still has to deal with certain phenomena of the sensations of sound and of light, which logically belong to physiology or psychology; but there is comparatively little difficulty in so isolating these portions of the text from those which may be dealt with in a purely objective way that there need remain no reason for the mental confusion which so often arises from the natural tendency to accept a sensation as a just measure of its awakening cause. Thus, if we set aside Chapters XXXIV and XLIV of the following pages, which treat primarily of the relations of sensations to their physical causes, and which will doubtless long be wanting in the scientific precision that characterizes other portions of physics, we find that all the remainder can be described as a strictly quantitative study of various transferences and transformations of energy. An understanding of energy is, therefore, absolutely essential to a satisfactory intellectual grasp of physics. This can only be attained by sustained study of dynamics, whence elementary mechanics must be regarded as the logical basis of the whole science of physics. No pains should be spared on the part of the student in attaining clear notions on this portion of his course. This conviction has prompted the writers to make

their treatment of mechanics more complete than is ordinarily the case, especially in the physical notions which attach to the simplest cases of the action of forces. For the purpose of giving familiarity with these ideas, many problems are appended to the various chapters, and the utility of exercises which involve the solution of these by the students is strongly urged upon teachers.

The class of students for which this text-book is designed is supposed to have a useful knowledge of trigonometry, but not of calculus. This is in agreement with the courses of instruction in most of our American colleges; but it has ordinarily the disadvantage of leaving rather a large interval between the study of the philosophy of physics and the application of its principles to engineering. Especially true is this of thermodynamics and electricity where it is often difficult for the student to recognize the fact that the unaccustomed mathematical processes are simply easier means of attaining an understanding of a physical problem and not an end in themselves. For this reason the subjects mentioned are developed with somewhat more completeness than usual, so that the engineering student can find the essential notions of his advanced work logically connected with those acquired at an earlier time. Of course such an extension implies an exercise of choice on the part of the teacher as to what may be omitted in first reading with a class of which only a portion expects to pursue the subject farther.

In Chapter XLI will be found a treatment, quite elementary in character, of the limiting powers of optical instruments. This is, of course, of great philosophical interest, because it is by the means of such instruments that we attain the greatest enlargement of our intellectual horizon. Notwithstanding the simplicity of the exposition, it does not seem to have been done before in unmathematical language.

The book as a whole is designed as an aid to the teacher in presenting a general view of the phenomena and philosophy of physics; but it assumes as an essential complement a course of demonstrations in the form of experimental lectures, and, for those students who aspire to acquire a knowledge of physics more than merely sufficient to enable them to follow the growth of science with an intelligent interest, a supplementary course in the physical laboratory.

CHARLES S. HASTINGS.
FREDERICK E. BEACH.

SHEFFIELD SCIENTIFIC SCHOOL OF
YALE UNIVERSITY, January, 1899.

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GENERAL PHYSICS.

PART I. — MECHANICS.

CHAPTER I.

PHYSICAL QUANTITIES AND THEIR MEASUREMENT.

1. Measurement of Physical Magnitudes. — The measurement of any physical, *i.e.* concrete, magnitude, consists in the comparison of the quantity to be measured with a definite portion of the same magnitude selected as a unit. The process of comparison may be either direct or indirect. In practice, the direct method is used only in the comparison of lengths, and the measurement of all other magnitudes has, with an insignificant exception, been reduced to the judging of the coincidence of two lines or the determination of a length, for the reason that the perceptions of the eye are more trustworthy than those of any other sense. The result of the comparison of physical quantities is expressed as so many times the unit chosen. If, for example, a rod is found to be four times as long as a foot-rule, its length is said to be 4 [ft.]. The numerical part of this expression is termed the *numeric*, and the part enclosed in brackets the physical *unit*.

It is obvious that no estimate of the size of the quantity measured can be formed unless the unit is explicitly stated.

2. Reduction of Observations. — The numbers which arise from the measurement of continuous physical quantities differ in an important respect from those numbers relating to discrete quantities which are the subject of calculation in arithmetic. The latter are complete and exact; the former will always be incomplete and subject to error either on account of the limitations of the observer or the imperfections of the instruments by whose aid they are obtained. Hence, in order to obtain trustworthy results from a calculation, it is desirable to make a critical inspection of the numbers which enter into every computation. The importance of such examination may be made clear by an example. Suppose, for instance, that there are 124 piles of coin on a table, and that each pile contains 48 coins, making a total of $124 \times 48 = 5952$ coins. This number is complete and cannot be stated with greater accuracy.

On the other hand, suppose that it is desired to compute the area of a table top from its measured dimensions, which by the aid of a scale have been found to be the following:

Length on one side	96.33 cm.
" " opposite side	96.35 cm.
Width " one end	68.69 cm.
" " opposite end	68.65 cm.

Selecting the mean of each of these pairs as the most probable value, the length may be taken as 96.34 cm. and the breadth as 68.67 cm. It is manifestly absurd to express this number to thousandths of a centimeter, for if the table were to be measured at intermediate points the figure in the second decimal place would very likely be changed. To refine these measurements would, in general, be labor thrown away, as far as any practical use of this area is concerned. Accepting, then, as satisfactory a measurement which is accurate

enough for the purpose, let the doubt which attaches to the last figure of each number be denoted by the italic type and the calculation performed in the usual way. Thus:

$$\begin{array}{r}
96.3\mathit{4}^{\mathrm{cm}} \\
\underline{68.6\mathit{7}^{\mathrm{cm}}} \\
\mathit{67438} \\
5780\mathit{4} \\
7707\mathit{2} \\
\underline{5780\mathit{4}} \\
661\mathit{5.6678}^{\mathrm{sq\ cm}}
\end{array}$$

It will be noted that the doubtful figures have rendered worthless at least four figures of this product. In other words, the percentage error of a product cannot be less than that of the worst factor entering into it. In the example given, the last four figures of the product might better be omitted; for, as they stand, they are liable to mislead the casual reader into supposing the result more accurate than it is.

The labor of calculation of useless figures without sacrifice of any precision in the result may be saved by the following abridged method of multiplication. Reverse the order of the figures in the multiplier, writing them directly below those of the multiplicand. The multiplication may now proceed in the usual way except that the first, *i.e.* the right-hand, figure of each partial product is obtained by multiplying the figure of the multiplicand by the one directly below it in the multiplier, mentally adding anything which should be carried from the multiplication of the preceding figure of the multiplicand by this figure of the multiplier. Writing these partial products so that their right-hand figures stand in a vertical column, they may then be added and the decimal point determined by inspection thus :

$$\begin{array}{r}
96.3\cancel{4} \\
76.86 \\
\hline
5780\cancel{4} \\
7707 \\
578 \\
67 \\
\hline
6615.6
\end{array}$$

The precision of this method is as great as before, and in the example given there is a saving of six multiplications and the addition of three columns.

If the numbers are large, the pointing off will be assisted by expressing them in powers of ten (Art. 3) before performing the multiplication.

The corresponding abridgment of division consists in deleting a figure of the divisor in each partial division after the first, instead of bringing down a cipher. For example:

$$\begin{array}{r}
6\cancel{8}.\cancel{6}\cancel{7})6615.6(96.34 \\
61803 \\
\hline
4353 \\
4120 \\
\hline
233 \\
206 \\
\hline
27 \\
27
\end{array}$$

Calculations involving several numbers may be most readily made by the use of logarithms. In this connection it should be noted that four-place tables are sufficient for the ordinary purposes of engineering, navigation, and the work of the chemical and physical laboratory, since in most cases these are all that the accuracy of the data will warrant. Very few measurements in the branches mentioned admit a precision

greater than $\frac{1}{3}$ of one per cent; on the other hand, the error in a four-place logarithm will never amount to more than $\frac{1}{20}$ of this, and is usually far less.

3. Notation in Powers of Ten. — The measurements which have to be recorded in Physics differ enormously in their relative magnitudes, though never exceeding in number seven significant figures. Thus, for example, the wave length of yellow light (D_1) is

$$\lambda = 0.00005896 \text{ cm.},$$

while the velocity is

$$v = 29{,}990{,}000{,}000 \text{ cm. per sec.}$$

Instead of writing so many ciphers, the following brief and convenient notation in powers of ten is often used:

$$\lambda = 0.5896\ (10)^{-4} \text{ cm.}$$
$$v = 2.999\ (10)^{10} \text{ cm. per sec.}$$

It is convenient to place the decimal point before the first figure if it is greater than 5, and after if less, as in above example.

4. Fundamental Magnitudes. — Experience has shown that the measurement of all physical quantities may be expressed in terms of three fundamental magnitudes. Those commonly chosen for this purpose are time, length, and mass, or quantity of matter. It may be assumed that our ideas of time and space, whether innate or acquired, are sufficiently exact for all practical purposes. The case of matter, however, requires more particular consideration. Of the three magnitudes named, matter alone is directly cognizable by the senses, and invested with a variety of interesting properties. From the chemical standpoint the most important of these is its

$$\begin{array}{r} 96.3\mathit{4} \\ \underline{76.86} \\ 5780\mathit{4} \\ 7707 \\ 578 \\ \underline{\mathit{67}} \\ 661\mathit{5}.\mathit{6} \end{array}$$

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indestructibility, which may be stated as a formal proposition as follows: In any closed space the total quantity of matter is invariable, irrespective of the changes of form which it may undergo.

For present purposes matter may be defined as anything which can be weighed, and the quantity of matter as proportional to its weight, *i.e.* its attraction toward the earth.

[illegible]. Unit of Time. The unit of time used for scientific purposes is the second. It is defined as $\frac{1}{86400}$ part of the mean solar day, by which is meant the average time between successive passages of the sun across the meridian throughout the year.

The period of the rotation of the earth on its axis, as determined by successive transits of a star, is an interval of nearly perfect constancy, but of such length that it does not [illegible] in civil affairs, and hence is used only for astronomical purposes. It is known as the sidereal day.

[illegible]. Instruments for Measurement of Time. — Clock. The [illegible]

ment is a small spiral spring and balance wheel (Fig. 144), mounted so as to oscillate with the utmost freedom about a vertical axis with the bending and unbending of the spring.

The accuracy of a chronometer does not equal that of the pendulum clock at its best; but, on the other hand, the chronometer is capable of giving reliable results where, as at sea, the clock would be perfectly useless.

A watch does not differ essentially from the chronometer, except that it is smaller, and the mechanism of the escapement is altered to adapt it to the somewhat rougher handling it is likely to receive.

Chronograph. The chronograph (Fig. 1) is an instrument designed to record intervals of time. Its essential parts are

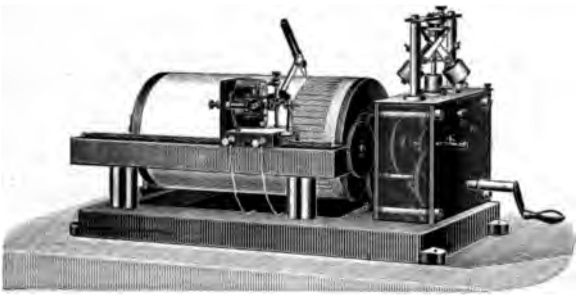


FIG. 1.

a cylinder carrying a sheet of paper and revolved uniformly by clock-work once a minute. Supported above the paper, and resting upon it so as to leave a helical trace as the cylinder turns, is a pen electrically connected with a clock, and so arranged that each time the clock beats it receives a slight and momentary displacement lengthwise of the cylinder. It is also capable of a similar movement at the will of the observer, by the depression of a telegraphic key. A copy of the record thus made is shown in Fig. 2.

The V-shaped notches mark the beginning of each second, while the square indentations record the epoch of two events at A and B observed by the operator. The interval which

Fig. 2.

elapsed is found by measuring the distance from A to B and comparing it with the scale of seconds.

In another form of chronograph the record is made on a sheet of smoked paper by a style fastened to the prong of a tuning fork of known period. The trace in this case is a sinuous curve on which the beginning and end of an event are marked by minute perforations, as at A and B, which are produced by an electric spark made to pass from the

Fig. 3.

style to the cylinder when the circuit is closed. Since each wave corresponds to a definite interval of time, the duration of the phenomenon may be readily deduced.

In the first form of instrument the limit of precision may be regarded as not far from $\frac{1}{100}$th of a second, while in the second it may be less than $\frac{1}{1000}$th of a second.

7. Unit of Length. — The unit of length employed in science is the centimeter. It is defined as one hundredth part of a certain platinum bar in the possession of the French Government, measured at a temperature of 0° C. This bar, the standard meter, was intended to represent a ten-millionth of the earth quadrant measured on the meridian of Paris from the equator to the pole; but as more recent measurements have revealed a slight deviation in the length of the bar from

this dimension of the earth, the meter is to be regarded in reality as an arbitrary rather than a natural length.

The relation of the centimeter to the inch is given very exactly by the equation

$$1 \text{ in.} = 2.5400 \text{ cm.}$$

8. Instruments for the Measurement of Length.—*Scales.* The determination of a length is commonly effected by the direct comparison of the required distance with a scale or copy of the standard of length, which is divided into equal parts by lines drawn on its face. The scale being

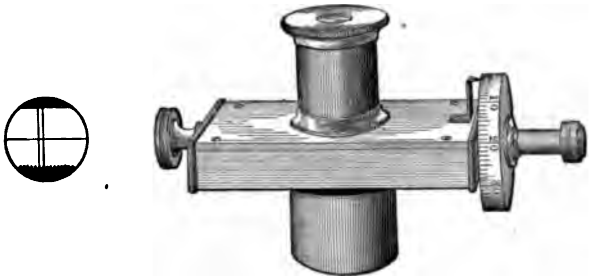


FIG. 4.

placed alongside the given length, the whole number of units may be read off and the fraction of a unit estimated to one tenth of a millimeter under favorable conditions. When greater accuracy is desired, various special instruments are used, differing somewhat with the manner in which the length to be measured is defined.

Micrometer Microscope. When a length is determined by the distance between two carefully ruled lines, the fractional part of a scale division may be found by the use of a micrometer microscope (Fig. 4). This consists of an ordinary microscope furnished with a pair of cross hairs, which can be moved across the field by a screw of fine pitch, and

the number of revolutions of the screw read from a disc attached to its head and divided into equal parts. By causing the cross hairs to traverse the small length which it is desired to measure, viewed through the microscope, the whole and fractional number of turns of the screw may be counted and reduced to the unit of length by means of the determined value of one revolution.

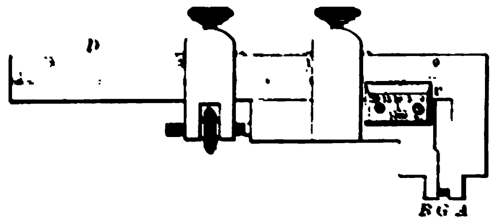


Fig. 5.

Vernier Caliper. When the length to be measured is defined by the bounding surfaces of a solid, use is often made of the instrument shown in Fig. 5, called a *vernier caliper*. It consists of a fixed jaw, *A*, and a movable one, *B*, sliding on the bar *D* and arranged so as to be brought into contact with the piece *G* to be measured. The opening of the jaws is then read off the graduated scale from the position of the point 0. To assist in the accurate determination of this point an auxiliary sliding scale, *V*, called a *vernier*, is added. Its principle is as follows: A distance, RS, is laid off on the sliding scale equal to nine divisions of the fixed one and divided into ten parts. If now the zeros of both scales be placed in coincidence, it is evident that 1 on the vernier will be one tenth of a scale division behind division 1 on the fixed scale. Similarly the divisions marked 2 on each scale will be two tenths scale divisions, 3, three tenths, and so on. Accordingly, to bring any mark, say 5,

Fig. 6.

of the vernier in coincidence with the corresponding one on the scale, the vernier must be moved six tenths of a division beyond the zero, or in general the number of the mark on the vernier in coincidence with one on the fixed scale gives the position of the vernier zero in tenths. Thus, in the figure the reading is 3.2. The precision of reading of the vernier is hardly more than twice that with which the eye will readily estimate the position of the line *R* on the scale *D*, but the convenience of its use is considerably in its favor.

Micrometer Caliper. The most accurate measurement of the linear dimensions of solid bodies of moderate size may be made by means of a micrometer caliper (Fig. 7). *A* is a fixed pin, and *B* the end of a screw which may be approached or withdrawn by turning the milled head of the sleeve *E*. The separation of *A* and *B* in any position is read by the aid of the scale on *D* and the graduated edge of the sleeve *F*. The precision of reading in such an instrument considerably exceeds the accuracy of setting.

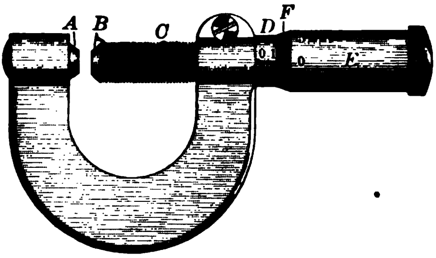


FIG. 7.

9. Unit of Mass. — The unit of mass used in science is called the gram. It is defined as an amount of matter equivalent to one-thousandth part of that of a certain piece of platinum, known as the standard kilogram, in the possession of the French Government. When this standard was constructed it was intended that the gram should be an amount of matter equivalent to that contained in a cubic

12. Dimensions. — When it is desired to make a general statement of a concrete magnitude without explicit mention of any unit, the numeric will be denoted by a small letter and the unit by a bracketed capital. Thus, any length may be written $l\,[L]$.

The most general expression for any derived unit is of the form

$$[L^p\; M^q\; T^r],$$

where the numbers p, q, and r are termed the ***dimensions*** of the derived unit in terms of the fundamental units. Numerous examples of dimensional formulae will be found in the articles following.

Since in any equation involving concrete magnitudes equality can subsist only between quantities of the same kind, it follows that every true equation must be homogeneous with respect to the physical units entering it. In this way dimensions furnish a valuable check upon the equations resulting from a train of reasoning. They are also of great assistance in changing from one set of units to another.

13. Derived Units. — *Area.* The area of any figure is proportional to the product of two of its linear dimensions. The dimensional formula of area is accordingly $[L^2]$, and the unit one square centimeter. This unit may be symbolized by 1 cm.²

Volume. The volume of any solid is proportional to the product of three of its linear dimensions. The dimensional formula of volume is $[L^3]$, and the unit one cubic centimeter. This unit is usually written 1 cm.³ or 1 cc.

Density. The density of a body, or the concentration of matter, is defined as the mass per unit volume. The dimen-

sions of density are ML^{-3}. The unit is the density of one gram per cubic centimeter. It is symbolized by $\frac{1 \text{ gm.}}{1 \text{ cc.}}$, but has received no name. The ratio of the density of any substance to the density of water is called the *specific gravity* of the substance. Since in the C. G. S. system the density of water is sensibly unity, there will be no occasion to use the term *specific gravity* unless the measures of mass or volume should be given in some other system.

Angle. The measure of an angle at the center of a circle is defined as the quotient of the subtending arc by the radius. The dimensions of angle are zero; that is to say, its measure is a pure number. A similar remark applies to all the trigonometric functions. The unit angle is an angle subtended by an arc equal to the radius. It is called the *radian.*

Velocity. When the position of a point changes continuously as time goes on, the point is said to possess a velocity, which is measured by the distance traversed per second, or the time rate at which its path is described. To illustrate, let s_0 (Fig. 9) be the position of the point measured along the path at the beginning of any time, and s_t its position at the end of t seconds. Then if equal distances are described in equal intervals of time, the velocity is constant and equal to the distance traversed divided by the time, or

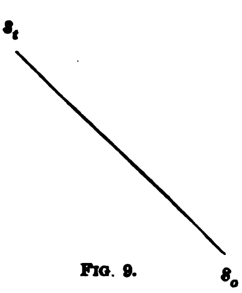


FIG. 9.

(1) $$v = \frac{s_t - s_0}{t}.$$

If the velocity is variable, its value at any instant will be the limit which this expression approaches as t is made smaller and smaller, or, in mathematical symbols,

$$(2) \qquad v = \text{limit}\ \frac{s_t - s_o}{t}.$$

The term *speed* is commonly used to designate the magnitude of a velocity considered apart from its direction. The dimensions of velocity are $\frac{L}{T}$ and the unit, a velocity of one centimeter per second. This unit is symbolized by $\frac{1 \text{ cm.}}{1 \text{ sec.}}$, but has received no generally accepted name.

Angular Velocity. When any line in a body changes its direction continuously with the time, it is said to possess an angular velocity, which is measured by the time rate at which the angle is described, or, in other words, by the angle swept over per second. When the angular velocity is constant, its value is found by dividing the angle traversed, by the time. As the dimensions of angle are zero, those of angular velocity will be T^{-1} and its unit one radian per second.

Acceleration. When the velocity of a point varies continuously with the time, the point is said to possess an acceleration which is measured by the time rate of change of the velocity. If v_o denote the velocity at the beginning, and v_t the velocity at the end of any time, t, the acceleration f may be written

$$(3) \qquad f = \text{limit}\ \frac{v_t - v_o}{t}.$$

If the acceleration is constant, this reduces to the change of velocity divided by the time. The dimensions of acceleration are LT^{-2} and the unit, an acceleration in which the velocity changes one centimeter per second, per second. This unit is symbolized by $\frac{1 \text{ cm.}}{1 \text{ sec.}^2}$, but has no accepted name.

Momentum. The quantity of motion, or momentum of a moving body, is defined as the product of the mass of the body by its velocity. Its dimensions are MLT^{-1}, and the unit, the momentum of a gram moving with a velocity of one centimeter per second. This unit is symbolized by $\frac{1 \text{ gm. } 1 \text{ cm.}}{1 \text{ sec.}}$, but has received no name. This important property of a moving body, which varies as both the velocity and the mass, may be illustrated by suspending two spheres of the same size but of different material, say wood and lead, by strings of equal length. If the spheres be set swinging together, it will be observed that an obstacle, such as a piece of moderately stiff paper, which largely affects the velocity of one, will produce but little change in that of the other. It is thus shown that, although the velocities of the two bodies appeared to be the same, the quantity of motion was greater in the greater mass.

Force. As a body has no power within itself to change its motion, we conclude in every case of changing motion that the body is acted upon by some external cause. This cause is called a force, and its measure is defined as the time rate of change of the motion. Thus, if F represents the force, mv_o the motion at the beginning of the time, and mv_t the motion at the end of the time, t, the expression for the force may be written

$$F = \text{limit} \frac{mv_t - mv_o}{t}. \tag{4}$$

Since m is constant, this may be written as the mass multiplied by the time rate of change of velocity, or

$$F = mf. \tag{5}$$

The dimensions of force are MLT^{-2}. The unit is that force which would change the velocity of one gram by one

centimeter per second, in a second. It is called a *dyne*, and is symbolized by $\frac{1 \text{ gm. } 1 \text{ cm.}}{1 \text{ sec.}^2}$. Experiment shows that a gram falling freely in a vacuum receives an acceleration of $980 \frac{\text{cm.}}{\text{sec.}^2}$; therefore, the attraction of the earth for the unit mass, or the weight of one gram, is 980 dynes. The acceleration due to weight is usually denoted by the letter g, thus:

$$g = 980 \frac{\text{cm.}}{\text{sec.}^2} = 32.2 \frac{\text{ft.}}{\text{sec.}^2}.$$

The weight of any mass, m, is accordingly mg.

An arbitrary unit of force much used in some branches of science is the attraction of the earth for the unit of mass. In this sense it is customary to speak of a gram force, or a pound force. The use of such a unit is open to two objections: 1° the application of the term *gram weight* to two things so distinct as a mass and a force is a source of serious confusion; and 2° the weight of a gram is not a perfectly definite force even on the surface of the earth, since it varies slightly from place to place. (See Art. 30.)

14. Vectors. — A vector, or directed quantity, is one which requires a number and a direction to define it completely.

A scalar quantity is one which is capable of complete definition by a single numerical specification, and whose value does not in any way involve a direction.

The displacement of a point, represented by a straight line drawn from the initial to the final position of the point, is the simplest illustration of a vector. A velocity, a momentum, an acceleration, a force, a flow, an electric current, the magnetization of iron, an angular velocity, are other examples of vectors.

Mass, volume, density, speed, and quantity of work are examples of scalars.

Any vector magnitude may be represented by a straight line drawn in the proper direction, and having as many units of length as there are units of quantity to be represented.

In the case of an angular velocity, the vector direction is taken as that of the advance of a right-handed screw, when revolved the same way as the body.

15. Sum of Two Vectors. — It is a matter of experience that if p and q (Fig. 10) represent two vectors without limitation as to position, then the line AC, in the triangle ABC, drawn with the side AB equal and parallel to p, and BC equal and parallel to q, will represent in magnitude and direction the sum or combined effect of p and q. This proposition may be most readily verified in the case of velocities, where it is easily shown that if a body moving with an assigned velocity have another velocity impressed upon it, the movement in the first direction takes place as if that in the second did not exist. An illustration of such combination of velocities may be found in Art. 474.

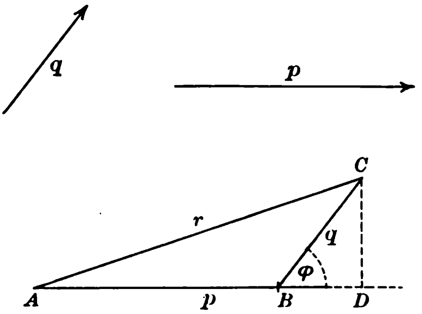


FIG. 10.

The magnitude of the sum of two vectors may be calculated as follows: From the point C (Fig. 10) drop a perpendicular on AB produced. Call the angle between the vectors, *i.e.* the angle through which q must be turned to coincide in direction with p, ϕ, and denote the length of AC by r. Then

$$(6)\qquad \overline{AC}^2 = \overline{AD}^2 + \overline{DC}^2, \quad \text{or}$$

$$r^2 = (p + q\cos\phi)^2 + q^2\sin^2\phi$$

$$(7)\qquad = p^2 + q^2 + 2pq\cos\phi.$$

16. Difference of Two Vectors. — If a minus sign before a vector be understood as reversing its direction, then the difference of two vectors, $AB - CD$, may be written $AB + DC$; that is to say, the difference is found by adding them after the reversal of the direction of the second vector. If the difference of two vectors, p and q (Fig. 11), be denoted by s, and the angle between them by ϕ, the magnitude of s is given by the relation

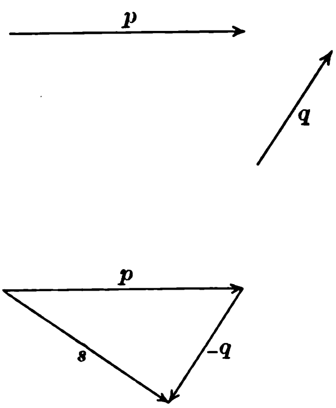


FIG. 11.

$$(8)\qquad s^2 = p^2 + q^2 - 2pq\cos\phi.$$

17. Resolution of a Vector into Components. — The resolution of a vector into components is not determinate unless the directions of the components required, or the

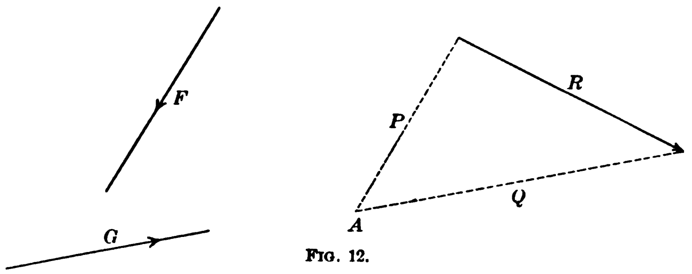
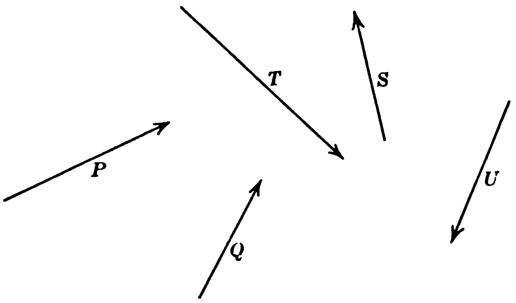


FIG. 12.

magnitude of one component and the direction of one, are given. For instance, let R (Fig. 12) be a vector which it is

desired to resolve into two components in the directions F and G. Through the extremities of R draw lines parallel to F and G which intersect at A, then P and Q are the components required, for $P + Q = R$.

The most important case is where the given directions are at right angles. If a be the angle between the given vector and one of the directions, the component parallel to this direction would be $R \cos a$, and the one at right angles $R \sin a$.



18. Sum of any Number of Vectors. — The resultant of a number of vectors, P, Q, S, T, U, may be found by first adding any two of them, then adding a third to this sum, and so on, as in Fig. 13. It is easy to show by trial that the final sum is independent of the order in which the vectors are taken; whence, as appears from Fig. 13, if the beginning of the second vector be applied to the end of the first, the beginning of the third to the end of the second, and so on, the line R drawn from the beginning of the first to the end of the last will represent in magnitude and direction the sum of the vectors P, Q, S, T, U.

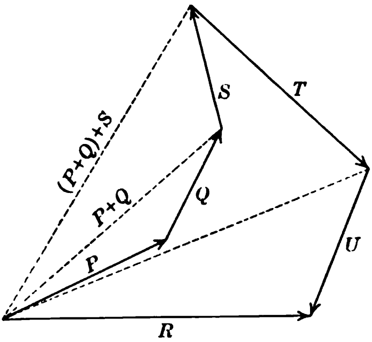


FIG. 13.

The magnitude of the resultant of any number of vectors is most easily calculated by referring the vectors to coördinate axes. Thus, let the lengths of the given vectors be denoted by $p_1, p_2, \cdots p_n$, and the angles which they respectively make with the axis of abscissas, by $a_1, a_2, \cdots a_n$. Call the horizontal components of the vectors $x_1, x_2 \cdots x_n$, and the vertical components $y_1, y_2 \cdots y_n$. Then the sum of the x components is

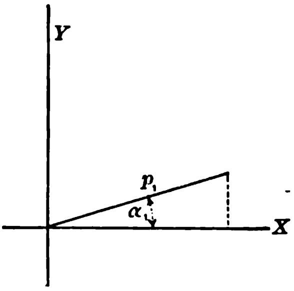


Fig. 14.

$$x_1 + x_2 + \cdots + x_n = p_1 \cos a_1 + p_2 \cos a_2 + \cdots p_n \cos a_n$$

(9) $$= \Sigma\, p \cos a = X, \text{ say.}$$

Similarly,

(10) $$y_1 + y_2 + \cdots y_n = \Sigma\, p \sin a = Y, \text{ say.}$$

The magnitude of the resultant R is

(11) $$R = \sqrt{X^2 + Y^2},$$

and the angle ϕ which it makes with the axis of abscissas given by

(12) $$\tan \phi = \frac{Y}{X}, \text{ or } \cos \phi = \frac{X}{R}.$$

The preceding results may without difficulty be extended to space of three dimensions. It is obvious that the difference of any number of vectors does not call for explanation since it may be reduced to the case of successive addition.

EXAMPLES.

1. A train runs 45.6 miles in 38.2 min. What is the velocity in C. G. S. units? $v = 3200$ cm. / sec.

2. The mean distance of the earth from the sun is $0.929(10)^8$ miles. What is the orbital velocity of the earth?
$v = 18.5$ miles / sec. $= 2.97(10)^6$ cm. / sec.

3. What is the velocity of a point on the equator, due to the earth's rotation, taking the sidereal day as four minutes shorter than the mean solar day? $v = 4.64(10)^4$ cm. / sec.

4. What is the angular velocity of the earth's rotation?
Ans. $0.729\,(10)^{-4}$ radian / sec.

5. What quantity of motion is possessed by a mass of 53.9 kilos moving with a velocity of 683 cm. per sec.?
Ans. $3.68(10)^7$ gm. cm. / sec.

6. The velocity of a body is observed to increase uniformly from 428 cm. per sec. to 1635 cm. per sec. in 13.0 sec. What is the acceleration?
$f = 92.8$ cm. / sec.^{2}.

7. What would an acceleration 986 ft. per minute, per minute, be in the C. G. S. system? $f = 8.34$ cm. / sec.^{2}.

8. If the mile be taken as the unit of length and the day as the unit of time, what will be the value of g? $g = 4.55\,(10)^7$ miles / day^2.

9. A constant force acting on a mass of 6283 gms. for 2.365 sec. changes its velocity from 5197 cm. per sec. to 9858 cm. per sec. Required the acceleration and the force.
$f = 1971$ cm. / sec.^{2}; $F = 1.238(10)^7$ dynes.

10. A rectangular block of stone containing 53.5 kilos has the following dimensions: 12.1 cm. by 31.7 cm. by 45.9 cm. What is its density?
$\rho = 3.04$ gms. / cc.

11. A sphere has a diameter of 56.4 cm. and a mass of 65.2 kilos. What is its density? $\rho = 0.694$ gm. / cc.

12. A cylinder has a hight of 74.3 cm., a diameter of 23.4 cm., and a density of 1.42 gms. per cc. What is its mass? $m = 45.4$ kilos.

13. The mean density of the earth is 5.53 gms. per cc. What is its mass? $m = 5.97(10)^{27}$ gms.

14. A right circular cone whose hight is 218 cm. has a mass of 98.7 kilos and a density of 5.64 gms. per cc. What is the radius of the base?
$r = 8.75$ cm.

15. A tube 19.3 cm. long contains 2.54 gms. of mercury. What is its mean cross section? *Ans.* 0.00968 sq. cm.

16. A flask containing 51.8 gms. of salt when filled with alcohol weighs 214.2 gms. When filled with alcohol alone it weighs 182 gms. What is the density of the salt? $\rho = 2.14$ gms./cc.

17. A flask weighs 39.74 gms. when filled with water. When 40.37 gms. of shot are inserted and the flask filled with water the weight is 76.54 gms. What is the density of the shot? $\rho = 11.3$ gms./cc.

18. A vector drawn east has a length of 16 cm., and one drawn northeast a length of 25 cm. What is their sum? *Ans.* 38 cm.

19. A ball is thrown south with a velocity of 36.7 ft. per sec. from a train running at a velocity of 78.4 ft. per sec. in a direction south 30° east. What is the velocity of the ball with respect to the ground?
Ans. 112 ft./sec.

20. A particle is acted on by four forces whose magnitudes are 695 pounds, 872 pounds, 384 pounds, and 243 pounds, respectively. The angle between each force and the adjacent one is 90°. What is the resultant force? *Ans.* $F = 702$ pounds weight.

CHAPTER II.

SIMPLE TYPES OF MOTION.

19. Unchanging Motion.—The simplest kind of motion is unchanging motion, *i.e.* motion in a straight line with uniform velocity. The equations may be written down at once from the definition. Thus, the force is zero because the motion does not change, or,

$$F = 0.$$

The acceleration is zero because the velocity does not change, or,

$$f = 0.$$

Also, by definition, the velocity may be written

$$\frac{s}{t} = \text{const.} = u,$$

whence

$$s = ut. \tag{1}$$

These equations contain the complete solution of this type of motion. Two of the conclusions are in apparent contradiction to experience, namely, that a body set in motion goes on forever; and, secondly, that no force is required to keep a body moving uniformly. But this is only because we have no experience of an absolutely unhampered motion, that is, of a body acted on by no force whatever. In the case of a stone sliding on smooth ice, the conditions are obviously more nearly approached than in most instances of sliding, and the approximation to the ideal case of perfect freedom from extraneous forces is nearer. So, also, a heavy ship moving very slowly through still water loses its

velocity with extreme slowness. It is considerations such as these which have led to the recognition of the simple law given above.

20. Uniformly Changing Rectilinear Motion. — An important type of motion is that in which a body moves with uniformly changing velocity in a straight line. Since by definition the acceleration is constant,

$$f = \frac{v_t - v_0}{t}. \tag{2}$$

The force is also constant, and equal to the mass multiplied by the acceleration,

$$F = mf. \tag{3}$$

Since the velocity increases f units per second, the gain in t seconds would be ft units of velocity, which, added to the initial velocity, gives

$$v_t = v_0 + ft. \tag{4}$$

This is of course the same equation as (2).

The average value of the velocity of any motion is the total distance divided by the time. Denoting it by $\bar{v}$,

$$s = \bar{v}t. \tag{5}$$

In this type of motion the average velocity may be easily calculated, since the velocity increases uniformly from v_0 to v_t; thus,

$$\bar{v} = \frac{v_0 + v_t}{2} = v_0 + \frac{ft}{2} \text{ by (4).} \tag{6}$$

This result may be represented by a diagram. To do this, let the time be plotted as abscissa and the velocity as ordinate (Fig. 15); then evidently the average hight of the line AB is given by the middle ordinate, or that of the point C, which is one-half the sum of those at A and B; *i.e.*

$$v_0 + \tfrac{1}{2} ft.$$

Multiplying this value of the average velocity by the time gives for the distance traversed

$$s = v_o t + \tfrac{1}{2} f t^2 \tag{7}$$

By the method of derivation it is evident that the value of s is represented by the area between AB and the axis in Fig. 15.

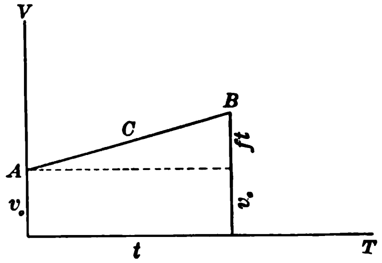


FIG. 15.

If t be eliminated between (4) and (7),

$$v_t^2 = v_o^2 + 2fs. \tag{8}$$

Summing up the result of this investigation, it is seen that whenever a constant force acts on a free body for a time, t, it increases its velocity by ft units, and increases its change of place by $\frac{1}{2}ft^2$ units of length, both of which effects are independent of the initial velocity which the body has.

21. Application to a Falling Body. — The most interesting instance of the type of motion just discussed is that of a falling body. If the body be supposed to start from rest, $v_o = 0$. Calling the acceleration g,

$$v_t = gt, \tag{9}$$

$$s = \tfrac{1}{2} g t^2, \tag{10}$$

$$v^2 = 2gs, \tag{11}$$

$$F = mg = \text{weight of } m \text{ grams.} \tag{12}$$

It will be noted that the distances traversed are as the squares of the times; that is, for 1 second, 2 seconds, 3 seconds, 4 seconds, the corresponding distances gone are as the numbers 1, 4, 9, 16.

Since the distance traveled in n seconds is proportional to n^2, and that in $n-1$ seconds to $(n-1)^2$, it follows that the

distance traversed in the nth second is proportional to $2n-1$, or the distances fallen through in the 1st, 2d, 3d, 4th seconds are, respectively, proportional to 1, 3, 5, 7, etc.

If a body be thrown vertically upward with an initial velocity, v_o, the upward direction being called positive, the equations of motion become

$$v_t = v_o - gt, \tag{13}$$

$$s = v_o t - \tfrac{1}{2}gt^2, \tag{14}$$

$$v_t^2 = v_o^2 - 2gs. \tag{15}$$

The time and hight of rise are found by setting $v_t = 0$, which is evidently its value, at the highest point. Thus,

$$t = \frac{v_o}{g},$$

$$s = \frac{v_o^2}{2g},$$

which, by comparison with (9) and (10), show that the time and hight of rise are the same as would be required for the body falling from rest to acquire the initial velocity v_o.

22. Atwood's Machine. — A freely falling body moves too rapidly to permit observations to be conveniently made upon it in any ordinary room. This difficulty is obviated in the apparatus shown in Fig. 16, known as Atwood's Machine. Two equal brass cylinders, CC', are suspended by a flexible cord which passes over the light pulley P, supported so as to turn with as little friction as possible. T is a swinging shelf on which C may be placed and released at a given moment, by a clock beating seconds. R is a ring-shaped stage with a hole large enough to permit C to pass freely through it. If the slip A be placed upon C when the latter is resting upon the shelf T, then when the clock beats, the system CPC' will be released and begin to move

under the influence of the weight of A, with an increasing velocity until C reaches the ring stage where A will be removed. From this point the motion will be uniform until C is arrested by the shelf S. Neglecting friction and other resistances, the acceleration of the first motion may be calculated as follows:

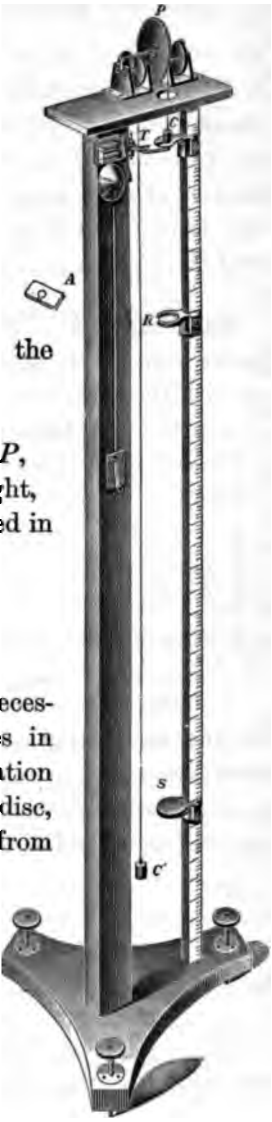


Fig. 16.

Let m denote the mass of each of the cylinders CC'.

Let m' denote the mass of the slip A,

M " " " " " pulley P,

g " " acceleration of weight,

f " " " produced in the system,

then, by equation 3,

$$\text{acceleration} = \frac{\text{force}}{\text{mass}}.$$

In applying this equation it will be necessary to add $\frac{1}{2}M$ to the other masses in order to take account of the acceleration of the pulley, supposed to be a flat disc, whose motion is of a different type from that under consideration. Thus, the total mass accelerated is $2m + m' + \frac{1}{2}M$, and the acting force is $m'g$; hence the acceleration will be

$$(16) \quad f = \frac{m'}{2m + m' + \frac{1}{2}M} g.$$

By a proper choice of masses, f may be given any value between

g and zero, and the system be made to move slowly enough for convenient observation.

If f be determined by observation from a particular set of masses, equation 16 may be used to calculate g, but the value so obtained will not be very accurate, on account of several sources of érror which are present. The greatest of these is the difficulty of determining the arrival of A at the stage R, within $\frac{1}{100}$ of a second.

23. Projectile. — Suppose that a heavy particle is projected into empty space with a velocity, u, and at an angle, α, with the horizontal plane.

Let the coördinates of the moving particle P (Fig. 17), at any instant, referred to axes passing through the starting point, be x and y. Then, since the only force acting on P is the downward force of weight, the motion parallel to the Y axis will be uniformly accelerated, but in the horizontal direction the motion will be a uniform one. If the velocities in these directions be denoted by v_y and v_x, respectively,

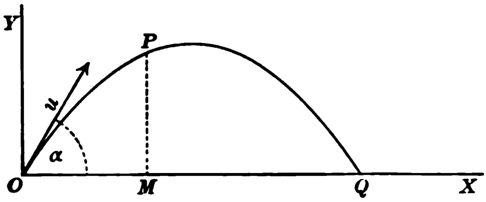


FIG. 17.

$$(17) \qquad v_x = u \cos \alpha,$$

$$(18) \qquad v_y = u \sin \alpha - gt,$$

by equation 13. Again, from equation 1,

$$(19) \qquad x = u \cos \alpha \cdot t,$$

and from equation 14,

$$(20) \qquad y = u \sin \alpha \cdot t - \tfrac{1}{2} g t^2.$$

Eliminating t between (19) and (20),

$$(21) \qquad y = x \tan \alpha - \frac{x^2 g}{2u^2 \cos^2 \alpha},$$

which, being an equation of the second degree, shows that the path of the body, or the trajectory as it is called, is a conic section, and, since it has but one infinite branch, it must be a parabola.

The time occupied by the projectile in passing from O to P is called the time of flight. Let it be denoted by T; then, since $y = 0$ at Q, equation 20 gives

$$(22) \qquad t = \frac{2u}{g} \sin \alpha = T.$$

The distance OQ is called the range, which will be denoted by R. Letting $t = T$ in equation 20, or $y = 0$ in equation 21,

$$(23) \qquad x = \frac{2u^2}{g} \sin \alpha \cos \alpha = \frac{u^2}{g} \sin 2\alpha = R.$$

It is evident that the range is a maximum when $\sin 2\alpha = 1$, *i.e.* for $\alpha = \frac{1}{4}\pi$ or 45°.

Suppose the projectile when thrown at the elevation α' has a range, R', then if

$$\alpha' = \tfrac{1}{2}\pi - \alpha,$$

$$R' = \frac{u^2}{g} \sin (\pi - 2\alpha) = \frac{u^2}{g} \sin 2\alpha = R,$$

or the range is the same for α and its complement.

Since $v_y = 0$ at the highest point, equation 18 gives the time of rise,

$$t = \frac{u}{g} \sin \alpha,$$

which is one-half the time of flight, as it should be. Substituting this value of t in equation 20, the hight of rise is found to be

$$y = \tfrac{1}{2}\,\frac{u^2}{g}\,\sin^2\alpha = \mathrm{H}, \text{ say.} \tag{24}$$

These equations are, of course, for a projectile moving in a vacuum; but the path of a slow-moving projectile in the air will not vary much from the parabola here described. For relatively high velocities, however, the change due to the resistance of the air may be very great, but the path will always fall within the parabola.

24. Uniform Circular Motion. — The characteristic properties of the force necessary to the motion of a particle in a circle with uniform speed may be determined by the following synthetic method of reasoning:

1°. There is a force because there is a change of motion. If there were no force acting, the particle would move in a straight line, with constant velocity.

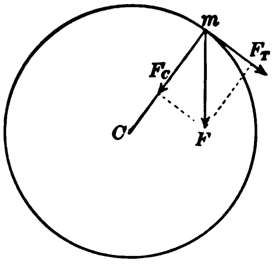


FIG. 18.

2°. The force is constant in magnitude because the change of direction is uniform.

3°. The force is directed toward the center; for suppose it had some other direction, F, as in Fig. 18, then this force could be separated into a central component, F_C, and a tangential component, F_T. Now the effect of the latter would be to increase the speed, which is contrary to the definition; therefore, there is no tangential component, and the force is all toward the center.

4°. The force varies as the mass, since by definition

$$F = mf.$$

5°. The force varies inversely as the radius of the circle; for suppose there are two circles, one twice as large

as the other, then, the speed being the same on both circles, in the same time the particle would describe equal arcs, AB and AB' (Fig. 19), but the change of direction, α, in going from A to B' would be twice that in going from A to B, or 2α. But the force is measured by its effect, that is, by the rate of change of the direction of the motion; therefore, the force is twice as great in the case of motion on the small circle, or,

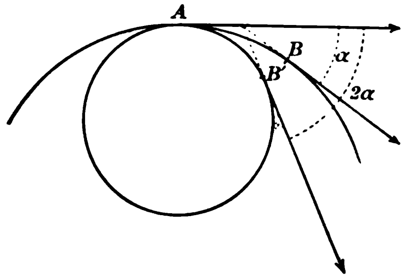


FIG. 19.

$$F \propto \frac{1}{r}.$$

6°. The force varies as the square of the speed; for suppose in one case the speed to be σ and in another case 2σ, then, if in the first case the particle moved from A to B (Fig. 20), in the second case it would go twice as far, AB', in the same time, or the change of direction in the second case is twice as great. But the quantity of motion, $2m\sigma$, in the second case is twice that in the first, $m\sigma$; therefore, the force in the second case changed twice the momentum through twice the angle, or the force must have been four times as great. Therefore,

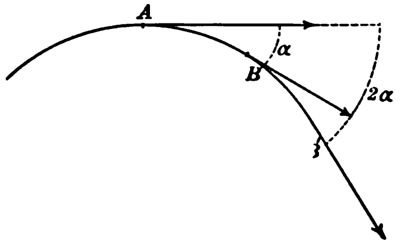


FIG. 20.

$$F \propto \sigma^2.$$

Collecting these results and denoting the central force by F_c,

$$F_c \propto \frac{m\sigma^2}{r}, \text{ or}$$

(25) $$F_c = \frac{km\sigma^2}{r},$$

where the value of k still remains to be determined.

The exact value of the acceleration in this type may be derived from a consideration of the rate at which the direction of the velocity is changed, as follows: Let the vector V_o (Fig. 21) represent the velocity when the particle is at A, and V_τ the velocity at the end of a small interval of time, τ. Also, let σ denote the speed, and α the angle AOB. Then, by the definition of acceleration,

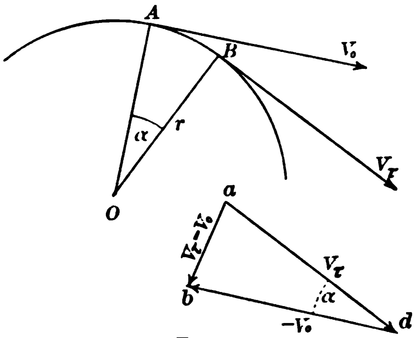


FIG. 21.

$$f = \underset{\tau=0}{\text{limit}} \frac{V_\tau - V_o}{\tau}.$$

Constructing the triangle abd according to the rule for subtracting vectors (Art. 16), it is seen that $V_\tau - V_o$ is a vector ab, having a direction which will be ultimately parallel to AO, *i.e.* toward the center of the circle. The magnitude of $V_\tau - V_o$ is from the same triangle,

(26) $$ab = 2ad \sin\frac{\alpha}{2} = 2\sigma \sin\frac{\alpha}{2},$$

whence, by substitution,

(27) $$f = \underset{\tau=0}{\text{limit}} \frac{2\sigma \sin\frac{1}{2}\alpha}{\tau} = \underset{\tau=0}{\text{limit}} \frac{\sigma\alpha}{\tau},$$

since the angle and its sine have the same limit. Observing that $AB = \sigma\tau$ and hence $\alpha = \frac{\sigma\tau}{r}$, the expression for the acceleration becomes

$$f = \frac{\sigma^2}{r}. \tag{28}$$

The value of the central force is accordingly

$$F_c = mf = \frac{m\sigma^2}{r}. \tag{29}$$

If the time of one revolution be denoted by P,

$$\sigma = \frac{2\pi r}{P}, \tag{30}$$

which in (29) gives

$$F_c = \frac{4\pi^2 mr}{P^2}. \tag{31}$$

These results may be obtained in a different manner by a consideration of the distance traversed under the action of a constant force in a straight line. Thus, suppose that the particle m moves with a discontinuous motion, being first drawn from A to B (Fig. 22) by a constant force, then moving from B to C with a constant velocity, to be again drawn a distance, CD, toward the center, and so on. It is obvious that by taking the steps of this path smaller and smaller, the motion

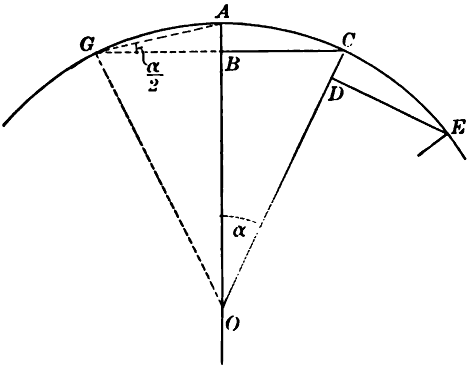


FIG. 22.

described may be made to approach uniform motion in a circle, without limit.

The distance AB is called the *sagitta* of the arc GAC. Let it be denoted by h, and let $AO = r$, $AOC = AOG = \alpha$, and $AG = a$. Then we may read at once from the figure,

$$\sin\frac{\alpha}{2} = \frac{h}{a} = \frac{a}{2r}, \tag{32}$$

whence

$$h = 2r\sin^2\frac{\alpha}{2} = \frac{a^2}{2r}. \tag{33}$$

Since by hypothesis the particle moved from A to B under the action of a constant force, the acceleration under these circumstances will be by equation 7, p. 27:

$$f = \frac{2s}{\tau^2}. \tag{34}$$

Substituting the value of s from equation 33 and passing to the limit, the acceleration for uniform motion in a circle becomes

$$f_c = \text{limit}\,\frac{4r\sin^2\frac{1}{2}\alpha}{\tau^2} = \text{limit}\,\frac{r\alpha^2}{\tau^2} = \frac{\sigma^2}{r} \tag{35}$$

on substituting the value of $\alpha = \frac{\sigma\tau}{r}$.

25. Conical Pendulum. — Suppose m (Fig. 23) is a heavy particle fastened by means of a string to a vertical axis, about which it is made to revolve. Let r be the radius of the circle in which the mass moves, and h the vertical distance from the point of attachment to the plane of the circle. Call the angle between the vertical and the string, θ. Now, since the mass is moving in a circle, it must be acted on by a central force, and this force must in some way be supplied

by its weight. Accordingly, suppose mg resolved into two components, one R, which stretches the string, and the other F_c, which is the central force required.

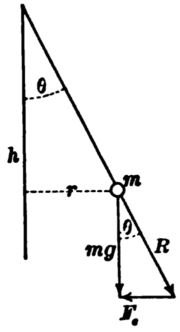


FIG. 23.

From the figure,

$$F_c = mg \tan \theta = mg\frac{r}{h},$$

and from equation 31,

$$F_c = \frac{4\,\pi^2 mr}{P^2};$$

equating and solving for P,

$$P = 2\pi\sqrt{\frac{h}{g}}, \tag{36}$$

which shows that the time of revolution is independent both of the length of the string and the mass of the bob. Hence, if various masses were fastened to a point of an axis by strings of different lengths, and the system were revolved in a definite period, the masses would all arrange themselves in the same plane, as seen in Fig. 24. The principle of the conical pendulum was utilized by Watt in his "governor," which has been a familiar feature on stationary engines till within a few years. In this apparatus two balls were used, one on each side of the axis, for the sake of balance, which by their change of hight with increase of speed of rotation were made to shut off the steam or admit more when the speed fell below a certain amount. Its action is not, however, sufficiently prompt to insure even moderate steadiness of running when applied to the throttle.

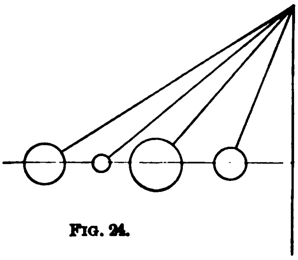


FIG. 24.

26. Parabolic Governor. — A more sensitive form of governor is that devised by Huyghens, in which the revolving masses were constrained to move on the surface of a paraboloid of revolution. The characteristic feature of this surface may be derived from the consideration of the form assumed by the surface of a liquid contained in a vessel and rotated about a vertical axis.

Let Fig. 25 represent a vessel containing a liquid which is rotating about the axis AA', and suppose that m is the mass of a particle of the liquid at the surface. In order to supply the necessary central force suppose the weight mg is resolved into a component, F_c, which is sufficient to secure the motion of the particle m in a circle of radius, r, and a component, N, which is counterbalanced by the reaction of the vessel transmitted through the liquid. When the particle m is in equilibrium the direction of N must be normal to the surface, for otherwise the particle would move either up or down. Thus, from the figure,

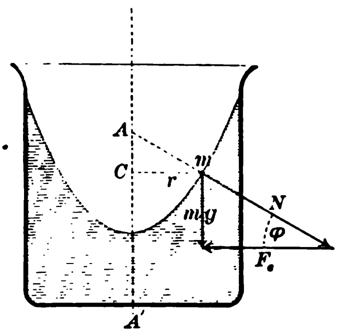


Fig. 25.

$$mg = F_c \tan\phi = \frac{4\pi^2 mr}{P^2} \tan\phi, \tag{37}$$

whence

$$r \tan\phi = \frac{P^2 g}{4\pi^2} = \text{const.} = S. \tag{38}$$

Now $r \tan\phi$ is the subnormal AC of the point on the curve, whose value is thus shown to be independent of the position of the point. As this is a well-known property of the parabola, it follows that the surface assumed by the liquid is a paraboloid of revolution.

Suppose now a particle, m, such, for instance, as a shot, placed on the surface of a paraboloid of revolution (Fig. 26), which is being whirled rapidly about the vertical axis. If the weight mg be resolved into a central force, F_c, and one normal to the surface, N, the particle will remain at rest with respect to the surface, for any definite period determined by

$$(39) \qquad P = 2\pi\sqrt{\frac{S}{g}}$$

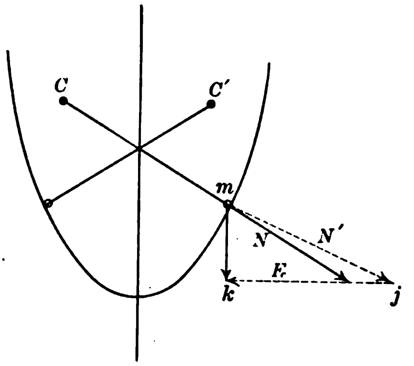


FIG. 26.

where S is the constant subnormal of the surface. If, now, the period of revolution be diminished, a greater value of F_c will be required, which may be obtained by resolving mg into F'_c, having a length, jk, and N', having a length, mj. But N' may be replaced by a component parallel to N, and one perpendicular, *i.e.* parallel to the surface, and the effect of this latter will be to slide the particle up the surface, without limit. By similar reasoning, it may be shown that, if the period was increased, the particle would slide down to the bottom. Hence, if the time of revolution differs from the value of the period required by the particular surface used, the particle can have no position of equilibrium on it.

In order to realize in a simple manner the condition that the balls of the governor should swing in a parabolic arc, Huyghens found it sufficient to attach each of them by a suitable arm to a point C or C' (Fig. 26), which is the center of curvature of the paraboloid at the place where it is desired that the ball shall be in equilibrium under the rotation.

27. Drying Machines. — Another important application of the principle of rotary motion is that made in the so-called centrifugal drying machines, which have come into extensive use in recent years.

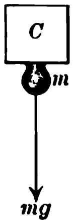


Fig. 27.

In order to compare the efficiency of draining and whirling, let m (Fig. 27) be a drop of liquid adhering to the solid C. The separating force will here be the weight of the drop, namely,

$$W = mg.$$

Next, suppose that C with the attached drop is whirled about the axis AA' (Fig. 28) in a period, P. Then, in order that m shall move with C in a circle of radius, r, the attachment of the drop to the solid must sustain a force

$$F_c = \frac{4\pi^2 mr}{P^2},$$

which is the measure of the separating force in this case. The efficiency of whirling relative to draining is accordingly

$$e = \frac{F_c}{W} = \frac{4\pi^2 r}{P^2 g}. \qquad (40)$$

If, for instance, r were 1 foot and P $\frac{1}{40}$ of a second, that is to say, if the body were revolved 2400 times per minute, the comparative efficiency of whirling would be 2000.

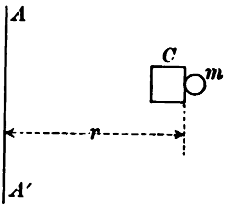


Fig. 28.

Some of the cases in which a centrifugal dryer is very useful are worth noting. The old method of drying clothes in the laundry was to wring or squeeze from the cleansed fabric as much water as possible, and let the rest evaporate by suspending so that the air would circulate freely about it. The modern way is to place the wet cloth in a cylinder with

perforated sides and whirl it rapidly, with the result that the cloth is dried far more thoroughly than is possible by wringing.

Another application is to the separation of sugar crystals from the syrup in the process of refining. Thus certain of the products of the refineries are called "centrifugal sugars." This method is also applied to the separation of oil from metal chips or waste. The modern automatic machinery used for working iron and steel, *e.g.* screw machines, or those used for drilling gun barrels, requires that the cutting tool be deluged with oil. The recovery of this oil from the refuse is an important item in the economical running of a large factory.

28. Cream Separator. — The same principle may be used to separate two liquids of different densities, provided neither acts as a solvent on the other. New milk, for example, may be regarded as an emulsion of fat and a liquid having nearly the density of water. If it be allowed to stand, the little globules of fat slowly rise to the top on account of their lesser density. If, however, the fresh milk be whirled in a suitable cylinder, the cream and the milk will rapidly separate into two co-axial layers, with the milk on the outside. To determine the relative efficiency of the two methods, let

m denote the mass of a small particle of cream,
V " " volume of a small particle of cream,
D " " density of the cream,
m' " " mass of the same bulk of milk,
D' " " density of the milk.

Then, if the particle of the milk and the cream be revolved about AA' (Fig. 29) in a circle of radius, r, the force neces-

sary to hold the cream in this circle will be

$$F_c = \frac{4\pi^2 mr}{p^2} = \frac{4\pi^2 r}{p^2} VD,$$

which must be transmitted through the liquid as a reaction from the side of the cylinder CD. Likewise the central force required for the milk will be

$$F'_c = \frac{4\pi^2 m' r}{p^2} = \frac{4\pi^2 r}{p^2} VD',$$

which must be greater than F_c, since D' is greater than D. Accordingly, if the pressure at the distance r is just sufficient

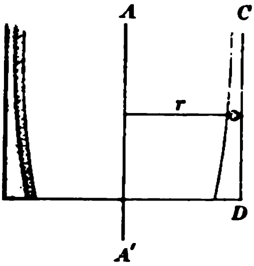


FIG. 29.

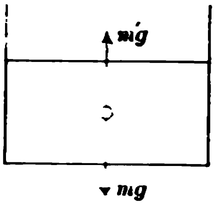


FIG. 30.

to hold the cream in that circle, it will not be great enough to retain the particle of milk, which accordingly takes up a position further from the axis. The separating force in this case is

$$(41) \qquad F'_c - F_c = \frac{4\pi^2 r V}{p^2}(D' - D).$$

The separating force in the case of pan setting may be obtained by the aid of the principle first annunciated by Archimedes, which asserts that, under the conditions assumed, the cream will lose a portion of its weight equal to the weight of its own bulk of milk. Hence the separating force (Fig. 30) may be written

$$(42) \qquad mg - m'g = Vg\,(D - D').$$

Dividing (41) by (42),

$$e = \frac{4\pi^2 r}{P^2 g}. \tag{43}$$

Fig. 31 shows one of the forms which the cream separator takes in practice. The milk is fed into the revolving bowl through the central tube *F*. The skim-milk is ejected into a chamber at *M*, and the cream expelled into a separate chamber at *C*.

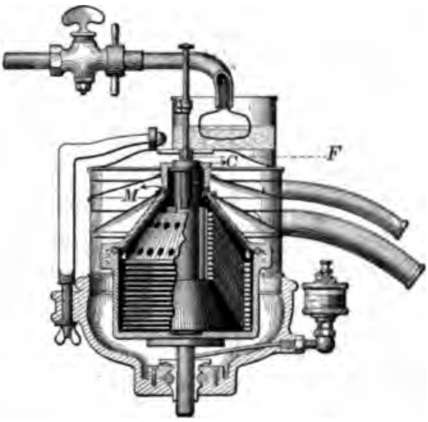


FIG. 31.

29. Law of Universal Gravitation. — The Law of Universal Gravitation, discovered by Sir Isaac Newton toward the end of the seventeenth century, may be stated as follows:

Every particle of matter in the universe attracts every other particle with a force whose direction is that of the line joining the two particles considered, and whose magnitude is directly as the product of the masses, and inversely as the square of the distance between them.

Expressed in symbols the law is,

$$F = G\frac{mm'}{r^2}, \tag{44}$$

where m and m' are, respectively, the masses of the particles, r the distance between them, and G a constant to be determined by experiment.

As a groundwork for this great generalization, Newton employed the results of two of the greatest astronomers who

preceded him, Copernicus and Kepler. About 1500 A.D. Copernicus perceived and announced that the apparent rotation of the heavens about the earth could be explained by supposing the earth to rotate on an axis once in twenty-four hours. Previous to this time the earth had been regarded as the center of the universe. He also showed that nearly all the motions of the planets, including the earth, could be explained on the assumption that these bodies revolved in circular orbits about the sun, whose position in the circle, however, was slightly excentric. The reason assigned for a circular orbit is a curious commentary on the attitude of the philosophers of that day toward nature. They said, it is manifestly improper that the heavenly bodies should move in any but perfect curves; the circle is the only perfect curve; therefore, the heavenly bodies move in circles.

Near the close of the sixteenth century Kepler, a German astronomer, began a study of the observations which had been made by Tycho Brahe, for the most part on Mars, for the purpose of ascertaining the laws of planetary motions. After twenty years of patient labor, with no method but that of trial and error, he succeeded in deducing the following laws which bear his name:

I. Each planet describes an ellipse in which the sun occupies one focus.

II. The radius vector (*i.e.* the line which joins the center of the sun and that of the planet) describes equal areas in equal times.

III. The square of the period of any planet is proportional to the cube of its mean distance from the sun.

Various surmises were made as to the explanation of these laws, but Newton was the first to prove the law of the force which would account for the motions of all the bodies in the solar system. Having first proved that the mass of

any approximately spherical and homogeneous body might be treated as if condensed at its center, he was able from Kepler's second law to show that each of the planets is kept in its orbit by a force constantly directed toward the sun. From the first law he showed that this force must vary inversely as the square of the distance from the sun.

From the third law he concluded that every planet behaved exactly as any other would do if substituted for it, that is to say, the force did not depend on the kind of matter composing it. In other words, the constant G was the same for all bodies. Being now satisfied that he had found the true law of gravitation as far as the planetary motions were concerned, he endeavored to test its generality by experiment. Assuming that the moon is held in its orbit by the same force as that which causes a body to fall at the earth, he saw it was possible to verify the law of distances by the space fallen through in a second. Thus, in Fig. 32 let $AB = a$ be the distance traversed by the moon in one second; also, let $AC = h$ be the distance which it has virtually fallen toward the earth; then

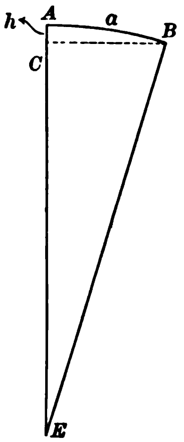


FIG. 32.

$$h = \frac{a^2}{2r},$$

by equation 33.

Taking the period of the moon as 27 d. 7 h. 43 m., and r as $3.844(10)^{10}$ cm., or sixty times the radius of the earth,

$$h = 0.1361 \text{ cm.}$$

The distance a body falls on the earth in a second is 490 cm., or practically 3600 times as far, which proves the law of the inverse square.

Newton's proof that weight depends only on quantity of matter, and not upon its kind, is explained in Art. 41.

If the law of gravitation be assumed, for the case of circular motion, which is very nearly that of planetary motion, Kepler's third law may be proved from it as follows:

Let M = mass of the sun,
m = " " a planet,
P = period of the planet,
r = distance of the planet from the sun;

then, by equations 31 and 44,

$$F_c = \frac{4\pi^2 mr}{P^2} = G\frac{Mm}{r^2};$$

whence

$$P^2 = \frac{4\pi^2}{GM} r^3 \text{ or } P^2 \propto r^3. \tag{45}$$

Equation 45 may be used to find the mass of the sun. Thus,

$$M = \frac{4\pi^2 r^3}{GP^2}. \tag{46}$$

The distance of the sun as determined by the aberration of light (Art. 543) is about 93,000,000 miles. The value of the gravitation constant G has been found with considerable precision by measurements with a torsion balance of the attraction between small metal balls. The experiments of Boys show its value to be

$$G = 6.657(10)^{-8}\frac{\text{cm.}^3}{\text{gm. sec.}^2}.$$

The mass of any planet which has a satellite may be found by an equation of the same form as equation 46. If the distance of the satellite from the planet be denoted by r'

and the period of the satellite by P', the mass of the planet will be

$$m=\frac{4\pi^2 r'^3}{GP'^2}. \tag{47}$$

Dividing equations 46 and 47, the ratio of the mass of the sun to that of any planet with satellite will be

$$\frac{M}{m}=\frac{P'^2 r^3}{P^2 r'^3}. \tag{48}$$

30. Diminution of Weight by the Rotation of the Earth. — To find the effect of the rotation of the earth upon weight, let

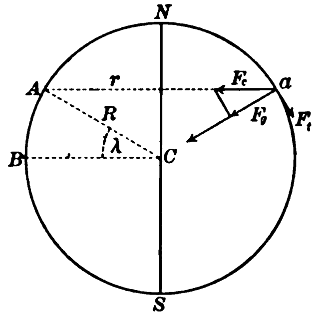


Fig. 33.

R (Fig. 33) denote the radius of the earth,

λ denote the latitude of any point, A,

r denote the distance of A from NS.

Then the central force necessary to secure the revolution of a mass, m, at this point in the period P of the earth, will be

$$F_c=\frac{4\pi^2 mr}{P^2}=\frac{4\pi^2 mR}{P^2}\cos\lambda. \tag{49}$$

At the equator the acceleration of this force will be

$$f=\frac{4\pi^2 R}{P^2}.$$

Substituting the numerical values of R and P,

$$f=3.40\ \frac{\text{cm.}}{\text{sec.}^2}=\frac{1}{289}g=\frac{1}{(17)^2}g;$$

from which it appears that if the period of the earth's rotation were reduced to $\frac{1}{17}$ of a day, since accelerations are

inversely as the squares of the periods, all bodies at the equator would lose their apparent weight.

Suppose that the portion of weight which is resolved so as to furnish the central force is denoted by F_g, and that its other component, perpendicular to F_g, is denoted by F_t, then

$$F_t = F_c \sin \lambda = \frac{4\pi^2 mR}{P^2} \sin \lambda \cos \lambda, \tag{50}$$

and

$$F_g = F_c \cos \lambda = \frac{4\pi^2 mR}{P^2} \cos^2 \lambda. \tag{51}$$

The former of these forces, considered as applied to the matter composing the earth, has produced a deformation in the shape of the earth, which is changed approximately from a sphere to an oblate spheroid having an equatorial diameter twenty-six miles greater than its polar diameter. The value of F_g is that portion of the total attraction of the earth which is requisite to secure the revolution of a body in a circle coincident with the surface of the earth. It follows from (51) that a body will appear to weigh more as it is carried toward the poles. The increment of apparent weight of a body, on account of the flattening of the earth in the polar regions, also varies very nearly as the square of the cosine of the latitude. Both of these influences on weight may be included in a formula thus: Let g_0 denote the acceleration of weight at the pole, and k, k' certain constants; then the value of the acceleration at any point may be written,

$$g = g_o - (k + k') \cos^2 \lambda;$$

or, by a simple transformation in the form,

$$g = a - b \cos 2\lambda. \tag{52}$$

The value of these constants as determined by observation

shows the value of g to be, in centimeters per second, per second,

$$g = 980.606 - 2.5028 \cos 2\lambda. \tag{53}$$

31. Influence of the Earth's Rotation on Projectiles. — The effect of the earth's rotation on a body projected horizontally in any direction at the surface of the earth is to deflect it toward the right in the northern hemisphere, and toward the left in the southern. Suppose, first, that a body is projected from B to A (Fig. 33). When it leaves B it will possess an eastward velocity equal to that of the earth, and greater than that of any of the points north of it, since B moves in a larger circle. Hence, as it passes these points, it will appear to be moving toward the east, that is, to turn to the right of an observer looking in the direction of projection. On the other hand, if a body be projected from a higher into a lower latitude, it will appear to lag behind the earth's surface in the latter region, since these points have a greater eastward velocity than the velocity of the point from which it started.

As has been shown in the previous article, a body moving with the earth is acted on by a component of weight, F_t, which urges it toward the equator. For any body resting on the earth this component may be regarded as inoperative, since the body would have, virtually, to ascend a slope thirteen miles in passing from a pole to the equator. If, now, such a body be projected toward the east, the central force to retain it in the circle must be greater than F_c. In order to provide for this there must be a different resolution of weight, and, in consequence, F_t will have a greater value. If, on the contrary, the body were thrown to the west, the values of both F_c and F_t would be reduced. In the first case the body, relative to one moving with the earth, would

be urged toward the equator with greater force, in the second case with less; or, in each case, the body would appear to turn to the right, looking in the direction of projection. Since, as has been shown, a body turns to the right when thrown in any one of four directions at right angles to each other, it follows that the body would be deflected to the right if thrown in any direction; for this latter direction may be resolved into components along two of the former taken as axes.

32. Cyclones. — The principle just deduced may be employed to explain the origin of cyclones and other circular storms. In the southern hemisphere their direction of rotation is always the same as the hands of a clock; in the northern hemisphere it is in the opposite direction. Let Fig. 34 represent a map of a portion of the earth's surface north of the equator, and suppose that for some reason the region A becomes more heated by the sun's rays than the surrounding portions. The air above this region will rise, since its density diminishes as its temperature increases. The cooler portions of air rushing in from the sides are deflected to the right, producing a whirl which travels across the country in a manner determined by the other, and much less clearly understood, meteorological conditions.

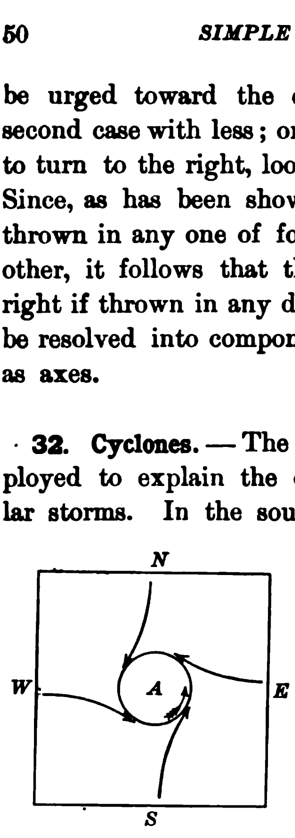


Fig. 34.

Violent atmospheric rotations on a small scale are called tornadoes. The dust whirls frequently seen on a windy day are little eddies in the air formed at the edge of some building or other obstacle.

33. Trade Winds.—Another illustration of the deflection of air currents, resulting from the revolution of the earth, is afforded by the trade winds which in low latitudes blow, in the northern hemisphere, with remarkable constancy from the northeast, and in the southern hemisphere from the southeast.

The air over the equatorial regions becomes heated and rises. The cooler body of air flowing in to take its place has a less velocity than the earth in this region, and hence is deflected toward the west with respect to the earth's surface.

The great ocean currents are a direct consequence of the prevailing direction of the wind in tropical regions, which produces a continuous drift of the surface of the water toward the west, giving rise to those great rivers in the ocean, such as the Gulf Stream in the Atlantic, or the Japan Current in the Pacific. These, too, illustrate the eastward drift of motion in the northern hemisphere, and strongly modify the climatic conditions of the northwestern portions of both continents.

34. Tides.—The alternate rising and falling of the waters of the ocean, known as the tides, is a complex phenomenon, depending upon the attraction of the sun and moon, and the revolution of the moon about the earth. The average time between alternate high tides is 24 h. 51 m., which is the same as the average interval between the successive passages of the moon across the meridian. The interval between the passage of the moon and the next high tide at any place is called the *establishment of the port.*

To explain why there are two high tides in a day, it is necessary to observe that the common statement that the moon revolves about the earth is not quite true, or is true only in the sense in which the earth may be said to revolve

about the moon. Both the earth and the moon revolve about an axis which passes through the center of mass of the system, *i.e.* a point which divides the distance between the centers of the moon and the earth inversely as their masses. Since these masses are about as 1 to 80, and their distance apart is 60 radii of the earth, the position of this axis must be about 3000 miles from the center of the earth. Let M (Fig. 35) represent the moon, E the earth, and AA the common axis of revolution, and consider what would happen if the earth were covered with water to a uniform depth. On the side of the earth next the moon the attraction of the latter operates to reduce the force of weight.

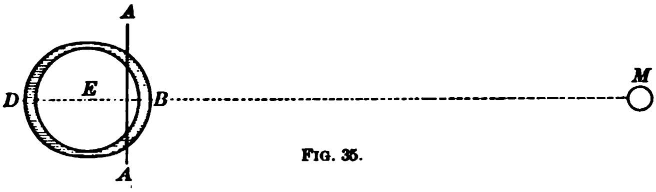


FIG. 35.

But the distance of the water on this side from the axis of revolution is also smaller than elsewhere, so that the water may be gathered together about B and still remain in equilibrium while revolving about AA under the influence of diminished weight. On the side opposite the moon a greater central force than elsewhere will be necessary to keep each particle of water revolving in a circle, since the distance from the axis is greater. But the attraction of the moon conspires with that of the earth on this side, so that it is possible to have the water, heaped up about D, revolve uniformly about the axis AA. Hence, as the earth rotates under the moon in but little over twenty-four hours, there will be two high tides in a day.

The action of the sun upon the water of the ocean is precisely similar to that of the moon, but on account of its

greater distance its tide-raising force is only two-fifths as great. At new and at full moon the solar and lunar tides unite, forming what is called a *spring* tide. When the moon is in quadrature the resultant is the minimum, or *neap* tide.

When the tidal wave strikes the shores of the continents, its energy is in large measure dissipated in the form of heat. As this energy is derived from that of the earth's rotation, it is evident that the ebb and flow of the ocean must result in a lengthening of the period of the earth's rotation.

It appears probable that the moon's day has been lengthened to the period of its synodic revolution by the operation of tides at a time when the moon was not a rigid body.

If the solid matter of the earth were as mobile as water, it would be subject to the same tidal waves as the ocean, and there would be no rise and fall of the tides with respect to the land.

If the solid portion were only a crust, the rise of the tides would be comparatively small with respect to the land, but their apparent hight would increase with the rigidity of the crust.

From reasoning founded upon the observed hight of tides, Lord Kelvin has drawn the conclusion that the interior of the earth must be for the most part solid, with a rigidity greater than that of glass.

35. The Gyroscope. — The circumstances of motion of an extended rigid body cannot be derived without the aid of more powerful mathematical methods than are furnished by elementary analysis; but there is one case which, on account of its interest and importance, may be noticed here. This case, namely, the one in which a rotating rigid body is constrained to move about a fixed point, is illustrated by the

gyroscope shown in Fig. 36. *R* is a massive ring revolving about an axis, *E*, which is carried by the frame *CFD*. The latter is pivoted at the fixed point *O*, and counterbalanced by a sliding weight, *B*. When the wheel *R* is at rest, the pivoted system behaves as any heavy bar would in the same circumstances. For instance, if the counterpoise be arranged so that the bar is horizontal and the standard *K* be rotated very slowly, the whole system may be turned with it as a result of the friction on the pivot.

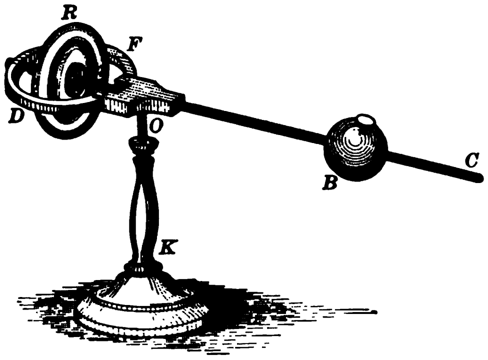


FIG. 36.

If, on the other hand, the wheel be set spinning rapidly, the system *CFD* will show a persistence of direction which is quite unaffected by the friction on the pivot when the latter is turned. If a force be applied to the bar to change its azimuth, the resulting change will be one of inclination, *i.e.* a change at right angles to the direction of the impressed moment.

If the counterpoise be shifted so that the system, when the wheel is at rest, is overbalanced, then when the wheel is set rotating the system will on the whole appear to revolve uniformly about a vertical axis through the pivot, though the actual motion is somewhat more complicated than this.

The relation of these several rotations may be most conveniently stated by representing them as vectors. Thus, let the rotation of the wheel *R* about the axis *E* be repre-

sented by the vector ω (Fig. 37), drawn parallel to EC. Also, suppose that the position of the counterpoise is such as to elevate the end C rotating the system about a horizontal axis perpendicular to EC, and let this rotation be represented by the vector θ (Fig. 37). Then the position of the vector ω', found by the addition of ω and θ, indicates the side toward which the axis of the gyroscope rotates about the pivot under the influence of weight.

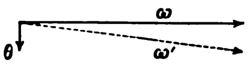


FIG. 37.

36. Precession of the Equinoxes. — It was discovered as early as 120 B.C. that the point where the sun crosses the celestial equator in the spring is moving continuously westward on the ecliptic. At the same time and for the same reason the celestial pole describes a circle 47° in diameter in the heavens.

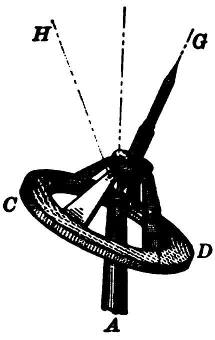


FIG. 38.

This phenomenon, known as the precession of the equinoxes, may be illustrated by the form of gyroscope shown in Fig. 38. CD is a massive ring attached by arms on one side to the spindle GB, whose end rests in a cup, B. If, when the ring is at rest, the axis be displaced from the vertical, it will oscillate through its position of equilibrium under the influence of weight. If, however, the ring be set spinning rapidly and then displaced, the axis will describe a vertical cone, BHG.

The conditions necessary for gyroscopic motion in the model, namely, a rapid rotation about an axis and the moment of a force tending to change the direction of the axis, are also fulfilled in the case of the earth rotating in space. On account of its departure from a spherical form, the earth

may be regarded as possessing a protuberant ring of matter in the region of the equator, which is inclined to the plane of the ecliptic (Fig. 39). Consequently, since the force of gravitation varies inversely as the square of the distance, it is evident that the attraction of the sun for the portion *D* must exceed that for the portion *C*, and will operate to restore the equator to the plane of the ecliptic. Since, however, this differential attraction of the sun is small, and the

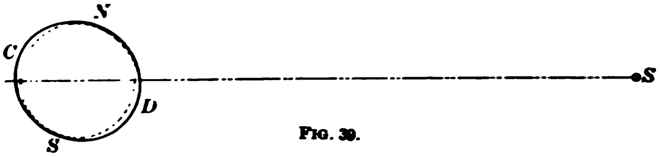


FIG. 39.

momentum of the earth due to its rotation is great, the precessional motion is very slow, a complete gyration of the axis occupying about 26,000 years. The present position of the celestial pole is about $1\frac{1}{4}°$ from Polaris; in 13,000 years from now it will be near Vega. The vernal equinox is at present about 30° from its first recorded position, consequently these observations must have been made something like 2000 years ago.

37. Nutation. — The uniform precessional motion of the earth's axis is subject to two disturbances: first, an annual variation in the moment of the force exerted by the sun, due to the change of position of the earth in its orbit; and, secondly, a variable moment exerted by the moon, with a period of about nineteen years. The effect of each of these is to impress a nodding motion on the earth's axis, so that the cone actually described is a fluted, instead of a smooth one. The influence of the moon in this phenomenon, called *nutation*, is several times greater than that due to the change in declination of the sun.

38. Rotating Projectiles. — In the case of a rapidly moving body the resistance of the air becomes a very important factor in the determination of the trajectory. A familiar illustration of this fact is seen in the erratic path of a light, irregular body, such as a shell when thrown from the hand. The influence of the air upon a projectile is usually prejudicial, though not universally, for the baseball pitcher and the boomerang thrower utilize it as the great essential of their art.

It must have been early discovered that a rotating missile could be thrown with surer aim, for arrows have been feathered for time out of mind. The rotation so produced practically eliminates the tendency to sidewise motion, which would be produced by any lack of symmetry in the shaft or bending which exists at the moment of release.

With the introduction of gunpowder the character of missiles underwent a complete change. The maximum destructive effect was now to be sought by making the mass and the velocity of the projectile as great as possible. As a bullet material, lead was found satisfactory, because both dense and comparatively cheap; but the problem of attaining a high velocity was not solved until the invention of the modern rifled arm. By the introduction of shallow, helical groovings in the barrel, a considerable rotation is imparted to the bullet, which prevents it from tumbling, and greatly lessens the liability to sidewise deflections. This construction permits the use of a longer and hence more massive bullet, without increasing the bore. The close fit required in the barrel also serves to prevent the escape of the imprisoned gases past the projectile.

The angular velocity impressed on the bullet in a modern rifle is comparatively large. For instance, if the rifling makes one turn per foot and the muzzle velocity of the

bullet is 1500 feet per second, the latter, after leaving the gun, will be rotating nearly 1500 times a second.

39. Harmonic Motion. — A particle moving with uniform speed in a circle, when considered with respect to its motion across a diameter, affords an example of an important type of motion, known as *harmonic motion.*

Let p (Fig. 40) be a particle of mass m, moving with constant speed, σ, in a circle. Draw any diameter, AB, and suppose that time is reckoned from the instant when the particle passes A. Let the distance dp of the particle from the diameter after a time, t, be denoted by x, and the corresponding angle at the center by θ. Then the component of the force F_c toward this diameter will be

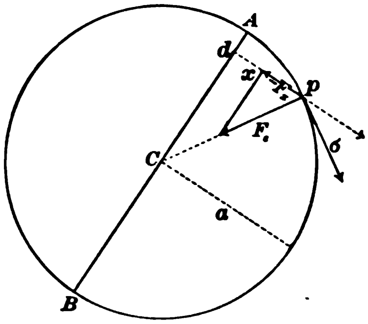


FIG. 40.

$$(54)\qquad -F_x = F_c \sin\theta,$$

where the minus sign denotes that the force is in the negative x direction. This force, since it is always opposite to the displacement of the particle, may be called the force of restitution. From the figure, calling the radius a,

$$(55)\qquad \sin\theta = \frac{x}{a};$$

whence, by substitution of the value of F_c from equation 31,

$$(56)\qquad -F_x = \frac{4\pi^2 mx}{p^2},$$

or,

$$(57)\qquad -\frac{F_x}{x} = \frac{4\pi^2 m}{p^2} = \text{const.} = k,$$

which shows that in harmonic motion the force varies directly as the displacement.

The velocity at right angles to AB is

$$v = \sigma \cos \theta. \tag{58}$$

Since the angular velocity is constant,

$$\frac{\theta}{t} = \frac{2\pi}{P}; \tag{59}$$

also, by definition,

$$\sigma = \frac{2\pi a}{P}.$$

Substituting these values,

$$v = \frac{2\pi a}{P} \cos \frac{2\pi t}{P}, \tag{60}$$

and

$$x = a \sin \frac{2\pi t}{P}. \tag{61}$$

It will be observed that the equations are now free from all distinctive reference to a circle, since P may be regarded as the period of vibration and a as the *amplitude*, *i.e.* the maximum value of the displacement.

Solving equation 57 for P,

$$P = 2\pi \sqrt{\frac{m}{k}}, \tag{62}$$

which, it may be noted, does not involve the excursion of the particle. Hence, harmonic motion may be defined as an oscillatory motion with respect to a line, in which the period is independent of the amplitude of vibration. The name is derived from the fact that the vibrations of musical sounds are of this type. That this is so appears from the fact that when a musical string is struck the pitch of the note does not change as the amplitude of the vibration decreases.

40. Simple Pendulum. — A heavy particle attached to a point by a string without sensible mass, and free to vibrate under the influence of weight, is called a simple pendulum.

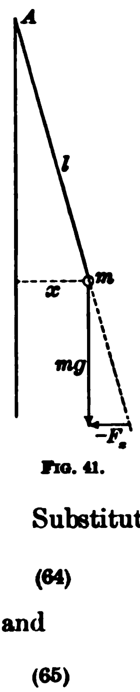


Fig. 41.

Let m (Fig. 41) denote the mass of the particle,
let l " " length of the string,
" θ " " angular displacement,
" x " " linear displacement.

Then, regarding the force of restitution as supplied by weight,

(63) . $$-F_x = mg \tan\theta.$$

If θ does not exceed about 3°, the angle

$$\theta = \frac{x}{l}$$

may be written in place of the tangent as a sufficient approximation.

Substituting these values in equations 57 and 62,

(64) $$k = \frac{-F_x}{x} = \frac{mg}{l},$$

and

(65) $$P = 2\pi\sqrt{\frac{l}{g}},$$

or the period of the simple pendulum, within the limits mentioned, depends only on its length and the acceleration of weight, and not upon the mass of the bob. The law of the pendulum, so far as it regards the relation of period and length, was first derived by Galileo, whose attention was directed to the problem by observing that a lamp suspended by a long chain in the cathedral of Pisa swung in a time which did not alter with the width of the excursion. From the independence of the period and the mass he drew the conclusion that, apart from the resistance of the air, a large

body would fall with the same velocity as a small one. This result he verified by dropping various bodies from the leaning tower of Pisa.

41. Proportionality of Weight and Mass. — The proportionality of weight and mass, though virtually assumed by every one who had used the balance, was first demonstrated by Newton. In order to investigate the relation between these quantities, he constructed a pendulum with a hollow spherical ball, in which he successively placed various different substances, and determined the period for each kind of matter. As no alteration in the period could be observed, he concluded that the attraction of one body for another depended only on the quantity of matter in each body, and not upon its nature.

42. Determination of g by the Pendulum. — The pendulum furnishes a very accurate method for the determination of the acceleration of weight by means of the equation

$$g = 4\pi^2 \frac{l}{p^2}. \tag{66}$$

In practice the length of the pendulum is conveniently chosen so that it beats seconds, since the period may then be easily determined from a standard clock by the method of coincidences. An observation having first been made as to whether the pendulum is gaining or losing on the clock, the time is noted when the beat of the pendulum and the clock first coincide, and again at the next coincidence, which may be several minutes later. If n be the number of seconds which elapse between two successive coincidences, the period will be $2\frac{n}{n \pm 1}$ according as the pendulum is gaining or losing. For the most accurate observations the form used is not the purely ideal simple pendulum, but a compound or

physical pendulum, consisting of a loaded bar which may be suspended from either of two adjustable knife edges, *L*, *M*, (Fig. 42). One of the cylindrical masses, *A*, is solid, while the other, *B*, is hollow, so that the distribution of matter is not alike in the two ends.

FIG. 42.

As will be shown in Art. 63, it is possible to find positions for these knife edges, unsymmetrical with respect to the center of gravity, such that the pendulum supported at either *L* or *M* will swing in the same period, which is also the period of a simple pendulum whose length is the distance between the points of suspension. This distance, defined by sharp metal edges, may evidently be determined with great precision.

43. Foucault's Pendulum Experiment. — In 1851 Foucault, a French physicist noted for his experimental ingenuity, devised a remarkable proof of the earth's rotation by means of a pendulum.

It is evident from theoretical considerations, or may be simply shown by experiment, that rotation of the support about the point of suspension of a pendulum will not affect the plane in which it is swinging, provided, of course, the mass of the bob is sufficiently great not to be appreciably affected by the resistance of the air. If, for instance, a pendulum be set swinging at the north pole in the prime meridian, this direction would be maintained unchanged, so that if a line should be drawn on the earth coincident in direction at the start, in twenty-four hours this line would have per-

formed a complete counter-clockwise revolution with respect to the plane of the pendulum. If the same experiment were to be tried at the equator, it is evident that there would be no motion of the earth with respect to the plane of vibration. The rate of rotation ω at any point whose latitude is λ may be found most simply by regarding the angular velocity of the earth as a vector, and taking its component in the direction of the vertical through the place, thus,

$$\omega = \frac{\pi}{12 \text{ hr.}} \sin \lambda.$$

Or it may be found as follows: Let PP' (Fig. 43) be the arc through which a point, P, rotates in a time, t, and let

R denote the radius of the earth,
r " " distance PH,
b " " " PN,
ϕ " " angle PHP',
ψ " " " PNP'.

Then, since the arc PP' is common,

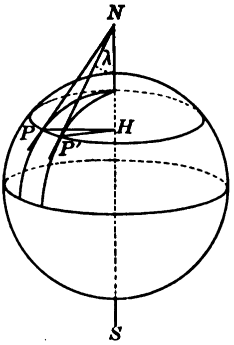


Fig. 43.

$$\frac{\psi}{\phi} = \frac{r}{b} = \sin \lambda\,; \tag{67}$$

whence the rate at which ψ is described is

$$\frac{\psi}{t} = \frac{\phi}{t} \sin \lambda. \tag{68}$$

But the angular velocity of P is $\frac{\phi}{t} = \frac{\pi}{12 \text{ hr.}}$, and therefore

$$\omega = \frac{\psi}{t} = \frac{\pi}{12 \text{ hr.}} \sin \lambda. \tag{69}$$

At a latitude of 45° this rotation is at the rate of 10.5 degrees per hour.

In starting the pendulum, considerable care must be exercised to assure its vibration exactly in a plane. This condition is best attained by drawing the pendulum to one side by a thread and burning it off when the system has come to rest. The experiment was first performed by Foucault in the Panthéon at Paris, by suspending an iron ball a foot in diameter by a wire 200 feet long. The result was in entire accord with his predictions, and was received with marked interest by the scientific world as furnishing a proof of the earth's rotation quite independent of astronomical observation. Belief in this rotation, it is true, had won universal acceptance before this time, but it was founded chiefly upon the improbability that the stars, which were known to be very distant, should move with a speed exactly proportional to their respective distances from the earth.

44. Examples of Harmonic Motion. — Other examples of harmonic motion are furnished by a strip of wood clamped at one end and loaded at the other with a mass, *m* (Fig. 44),

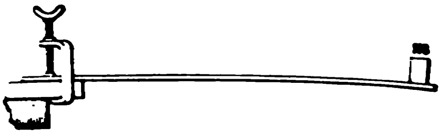


Fig. 44.


Fig. 45.

and by a helical spring attached to a bracket and weighted at the lower end, as in Fig. 45. If either of these systems be set vibrating, it will be found that the period is sensibly

the same whether the amplitude is a fraction of a millimeter or several centimeters.

If the displacement or elongation produced by m be denoted by e, the value of k will be

$$k = \frac{-F_x}{x} = \frac{mg}{e} = \frac{4\pi^2 m}{p^2}; \tag{70}$$

whence

$$g = \frac{4\pi^2 e}{p^2}. \tag{71}$$

The acceleration of weight, as found by observations on either of these pieces of apparatus, will in general be much less accurate than that found from a pendulum.

EXAMPLES.

1. A body falls freely from rest for 12.6 sec. Required the final velocity and the distance traversed. $v_t = 1.235(10)^4$ cm. / sec.
$s = 0.778(10)^5$ cm.

2. How long will it take a body to fall 650 ft., and what velocity will it acquire? $t = 6.35$ sec.
$v = 204$ ft. / sec.

3. A body is thrown downwards with a velocity of 874 cm. per. sec. Required its velocity and position at the end of 16.3 sec.
$v = 1.68(10)^4$ cm. / sec.
$s = 1.44(10)^5$ cm.

4. A body is thrown vertically upwards with a velocity of 827 cm. per sec. How high will it rise? $s = 349$ cm.

5. A body is thrown vertically upwards with a velocity of 697 cm. per sec. How long will it take to rise 195 cm., and what velocity will it then possess? $t = 0.383$ sec.
$v = 322$ cm. / sec.

6. A body is thrown downward with a velocity of 328 cm. per sec. What distance will it traverse during the 12th sec.?
Ans. $1.16(10)^4$ cm.

7. A ball is thrown to a hight of 150 ft. With what velocity does it leave the hand? $v = 98.3$ ft. / sec.

8. If a 5-ounce baseball be thrown with a velocity of 110 ft. per sec., and the hand traverse a distance of 3 ft. in imparting to it this velocity, required the force and the time of action, supposing this to be constant.

$F = 19$ pounds weight.
$t = 0.054$ sec.

9. Two bodies are dropped successively from the same point at an interval of 0.25 sec. When will they be 6.5 ft. apart?

$t = 0.68$ sec.

10. A body is projected upwards with a velocity of 965 ft. per sec.; 6.45 sec. later a second body is thrown up. When and where do they meet? $t = 26.8$ sec. after the departure of the second body.

$s = 14300$ ft.

11. A body is thrown vertically downwards with a velocity of 38.5 cm. per sec.; 2.57 sec. later a body is thrown after it with a velocity of 4750 cm. per sec. When and where will the second body overtake the first? $t = 1.52$ sec. after departure of second body.

$s = 8350$ cm.

12. A stone is thrown horizontally from a cliff, 364 ft. high, with a velocity of 105 ft. per sec. When and where will it strike the ground?

Time $= 4.75$ sec.
Distance $= 499$ ft.

13. A mass of 876 gms. is attached to a spring balance which is carried upward at such a rate that the balance indicates 932 gms. What is the acceleration of the motion?

$f = 62.7$ cm. / sec.^{2}.

14. A mass of 162 kilos hanging by a perfectly flexible cord drags a mass of 973 kilos along the top of a smooth table. What is the acceleration of the system, and what is the tension of the cord?

$f = 140$ cm. / sec.^{2}.
$T = 1.36(10)^5$ dynes.

15. Masses of 938 gms. and 762 gms., respectively, are hung by a flexible cord passing over a frictionless pulley. How far must the weights move in order to acquire a velocity of 325 cm. per sec.?

$s = 521$ cm.

16. What will be the tension on the string in the preceding example?

$T = 8.42(10)^5$ dynes.

17. A particle slides down a smooth plane 326 cm. long, inclined at an angle of 45° to the horizon. Required the time of descent and the velocity with which it will reach the bottom.

$t = 0.970$ sec.
$v = 672$ cm. / sec.

18. If a body fall 178.3 ft. in the 6th sec., what is the value of g?

$g = 32.42$ ft. / sec.^{2}.

19. Show that the time of descent on any chord passing through the highest point of a vertical circle is the same. $t = 2\sqrt{\frac{r}{g}}$.

20. A mass descending vertically draws an equal mass 25.3 ft. up a smooth plane, inclined at an angle of 30° with the horizon, in 2.5 sec. What is the value of g? $g = 32.4$ ft. / sec.^{2}.

21. The record for baseball throwing is about 400 ft. What was the velocity with which the ball left the hand?

$v = 113$ ft. / sec.

22. If a body be projected at an angle of 30° above the horizontal, from a cliff 80 ft. high, with a velocity of 96 ft. per sec., when and where will it strike the ground? Time = 4.2 sec.

Distance = 349 ft.

23. The ratio of the masses of the moon and the earth is 0.0125, and the ratio of their diameters is 0.273. With what acceleration would a body fall at the moon's surface? $f = 164$ cm. / sec.^{2}.

24. A mass of 53.8 gms. is constrained to move in a circle of 597 cm. radius with a speed of 235 cm. per sec. What is the force and what is the period of revolution? $F = 4.98(10)^3$ dynes.

$P = 16.0$ sec.

25. The distance of the moon is $3.84(10)^{10}$ cm., and the lunar month is approximately 27 d. 8 h. What is the acceleration of the earth's attraction at the moon? $f = 0.272$ cm. / sec.^{2}.

26. If a skater describe a circle of 98 ft. radius with a speed of 18 ft. per sec., what should be the inclination of his body from the vertical for equilibrium? *Ans.* 5.9°.

27. What must be the vertical distance between the mass and the point of suspension of a conical pendulum, in order that the period shall be 0.875 sec.? $h = 19.0$ cm.

28. The length of a conical pendulum is 45 cm., and the radius of the circle in which the mass moves is 12.6 cm. What is the period of the pendulum? $P = 1.32$ sec.

29. What would be the value of g if the period of a pendulum 97.31 cm. long were 1.975 sec.? $g = 984.9$ cm. / sec.^{2}.

30. If the period of a pendulum 99.7 cm. long is 2.12 sec., what will be the period of a pendulum 822 cm. long? $P = 6.09$ sec.

31. What is the length of a pendulum which loses 20 sec. per day, where $g = 980.3$ cm. / sec.^{2}? $l = 99.37$ cm.

32. What is the rate of a pendulum 39.09 inches long at the same place as the preceding example? Gaining 13 sec. per day.

CHAPTER III.

WORK AND ENERGY.

54. Work. — When a body moves in the direction of a force acting upon it, the force is said to do work, and the measure of the work is defined as the product of the force by the distance moved. If the direction of the displacement is inclined to the direction of the force, the work is found by multiplying the displacement by the component of the force parallel to the displacement. Thus, if the body A (Fig. 46) is displaced to A' a distance, s, under the action of the force F, making an angle α with this direction, suppose F to be resolved into a component, F', parallel, and a component, F'', at right angles to AA'. The work done by the component F'' will be zero, since the body suffers no displacement in its direction. Therefore, the total work will be that done by F'. Calling this work W,

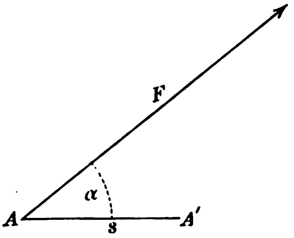


FIG. 46.

$$W = F' \cdot s = F \cos \alpha \cdot s. \tag{1}$$

The dimensions of work are ML^2T^{-2}. The unit is the work done by the force of one dyne acting through the distance of one centimeter. It is symbolized by $\frac{1 \text{ gm. } 1 \text{ cm.}^2}{1 \text{ sec.}^2}$, and is called the *erg*.

Units of work derived from the gravitational unit of force are sometimes used. Thus the kilogram-meter is the work done by a force equal to the weight of one kilogram acting

through one meter, and the foot-pound is the work done by a force equal to the weight of one pound acting through one foot.

46. Energy. — Energy is defined as the capacity of a body or system of bodies for doing work.

It is convenient to distinguish two kinds of energy: 1°, *potential* energy, or that due to the configuration of a system of bodies; and, 2°, *kinetic* energy, or the energy which a body possesses in virtue of its motion.

An example of each is furnished by a bow and arrow. When the bow is drawn, it possesses an amount of energy due to its altered shape, which is equal to the work done in bending it. When the string in contact with the arrow is released, the latter is given a rapid motion, in virtue of which it is able to do work when it is brought to rest.

The spring of a clock when wound up is another example of a system possessing potential energy, as is also a weight elevated a certain distance above the earth. The increase of potential energy of the system, which in this case consists of the weight and the earth, is equal to the work done in raising a mass, m, through a hight, h; that is, mgh. When the weight is allowed to fall to earth again, it will acquire an amount of kinetic energy which may be used to do useful work, such, for instance, as driving a pile into the ground.

47. Kinetic Energy of a Moving Particle. — It has been shown in Art. 20 that when a particle, m, is acted on by a constant force, F, for a time, t, the particle will suffer a displacement, $s = \frac{1}{2}ft^2$, due to the force, and acquire a velocity $v = ft$. Hence, the work done by the force during the process is

$$W = F \cdot s = mf \cdot \tfrac{1}{2}ft^2 = \tfrac{1}{2}mv^2. \tag{2}$$

This expression, or one-half the product of the mass by the square of its velocity, is taken as the measure of the kinetic energy of the moving particle.

The dimensions of energy are ML^2T^{-2}, the same as for work. The unit kinetic energy is the erg, or twice the kinetic energy of one gram moving with the velocity of one centimeter per second.

48. Transformations of Energy. — A vibrating pendulum presents an example in which the energy is continuously changing from the potential to the kinetic form, and *vice versa*. Thus, at the middle of the swing the energy is all kinetic. As the pendulum rises, the potential energy increases and the kinetic energy diminishes, until at the extent of the swing the former becomes a maximum and the latter zero. On the return to the lowest point the energy resumes its kinetic form.

The examples of energy mentioned so far have been of a purely mechanical sort. When a moving body is brought suddenly to rest by striking against an obstacle, the kinetic energy is mostly changed into heat; as, for instance, when a piece of lead is made hot by hammering it. When carbon and oxygen are allowed to unite, as in the burning of coal, the energy of chemical separation is likewise transformed into heat. When the diaphragm of a telephone is set vibrating, there is a transformation of the energy of mechanical vibration into the energy of electric currents in the wire, which are again transformed partly into the energy of vibrations at the receiver and partly into heat in the conductors. When a stick of resin is rubbed with fur, a part of the work expended is transformed into the energy of electrical separation. Many other examples could be given, but these are sufficient to illustrate the fact that energy not only appears

under a variety of forms, but also can be transmuted from one form into another.

49. Conservation of Energy. — The doctrine of the conservation of energy asserts that in any closed system, that is to say, one which is isolated so that it neither parts with energy from within nor receives energy from without, the amount of energy is invariable.

This principle, which may be regarded as not only the greatest but the most fruitful conception which has ever been introduced into physical science, represents rather the accumulated experience of all thinking men than the discovery of any individual or any school of philosophy. The first comprehensive statement of the law appears to have been published by Mohr in 1837, but many philosophers before his time had assumed its truth in special cases. As early as the end of the sixteenth century, Galileo, in his discussion of the laws of a falling body, recognized and used the principle that a body, in virtue of the velocity acquired in its descent, rose to the hight from which it had fallen, or would do so if the resistance of the air were removed. Huyghens went a step further, declaring that if a number of weights be set in motion the common center of gravity cannot possibly rise higher than the place it occupied when the motion began.

Newton, near the middle of the seventeenth century, perceived that the principle of conservation, which had hitherto been considered only in connection with bodies acted on by weight, was applicable to all mechanical problems. In the scholium to his Third Law he states that "if the action of an agent be measured by the product of the force into its velocity, and if similarly the reaction of the resistance be measured by the velocities of its several parts multiplied

into their several forces, whether they arise from friction, cohesion, weight, or acceleration, action and reaction in all combination of machines will be equal and opposite."

The quantity here defined is what would now be called the rate at which work is done by the forces, but for the field covered in the statement it is essentially equivalent to the assertion that energy is conserved.

By the beginning of the nineteenth century all instructed mechanicians had come to recognize that it was impossible to get any more work out of a machine than was expended upon it; and it is interesting to note that the French Academy, as early as 1775, declined to consider any further devices for securing "the perpetual motion," that is, machines which should not only keep running for an indefinite time, but also perform useful work.

At that time, before the nature of heat had been determined, it was not possible for any one to state whether the difference between the work supplied to a machine and the work derived from it was destroyed, or whether it had disappeared by passage into some other form of energy. The determination of the true nature of heat by the experiments of Rumford, of Joule, and of Davy, which will be explained more fully in Chapter XIV, gave the needed confirmation to the idea which had been more or less clearly apprehended by Carnot, Mayer, Joule, and others, that energy can neither be created nor destroyed. For having proved that a definite amount of work was always equivalent to a certain amount of heat, it was now possible to show that whatever form energy may assume it can always be made to yield an amount of work and heat, which together are equivalent to the work originally expended.

It is by experiments of this kind, tried in all sorts of ways, that the law has now been established, and it is recognized

to be universally true that whether the energy subsist in the potential form as the

energy of the visible arrangement of the parts of a system, or as the
energy of molecular separation, or as the
energy of electrical separation,

or whether it consist in the kinetic energy

of moving visible masses, or
of molecular vibrations, or
of wave motions in the ether, or
of electrical currents,

in each and every case the total quantity of energy in the system, when isolated from every other system, is invariable.

50. Minimum Potential Energy. — It has been found that when any of the stresses on a system are removed a redistribution of the energy occurs, that portion of the energy in the potential form assuming a minimum value and that in the kinetic form a maximum value. This principle, sometimes comprised in the statement that "potential energy tends to a minimum," is capable of application to a wide variety of problems. Thus, for instance, it furnishes a simple test for the three conditions of a system, known respectively as *stable*, *unstable*, and *indifferent* equilibrium. A system is in stable equilibrium if, in order to give it a small displacement, it is necessary to do work upon it, *i.e.* increase its potential energy. A right cone resting on its base presents an example of stable equilibrium. If, on the other hand, a slight displacement diminishes the potential energy, the system is unstable. A cone resting on its apex would be an example of unstable equilibrium. When a slight displacement produces no change in the potential

energy, the system is in indifferent equilibrium. This is illustrated by a right circular cone lying on its side on a horizontal surface.

51. Machines. — A machine is a piece of apparatus by the aid of which work may be more conveniently performed. In every machine there is some point to which energy is supplied, and another point from which work is derived. Usually the force exerted by the latter, or working point, is greater than that applied to the former. The ratio of the force derived to the force applied is called the *mechanical advantage* of the machine. This ratio may be expressed in terms of certain distances, which are more easily measured than the forces. Thus, let A (Fig. 47) be the point of a machine at which a force, F, is applied, and B the working point. Let D be the distance moved by the point of application of the force F, and D' that moved by the working point which exerts a force, F'. Then, if the machine is a perfect one,

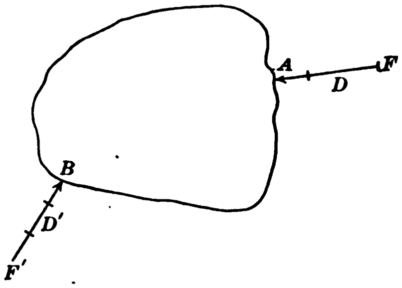


FIG. 47.

$$W = FD = F'D',$$

(3) or, the mechanical advantage $= \dfrac{F'}{F} = \dfrac{D}{D'}$.

The mechanical advantage may thus, without knowledge of the intervening mechanism, be calculated from an observation on the movements of the working point and the point at which the force is applied.

The following examples illustrate how the mechanical advantage of a perfect machine may be calculated when the arrangement of its parts is given.

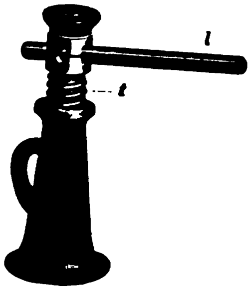


FIG. 48.

Screw. Let t be the pitch of a screw, that is, the distance between the threads, and l the length of the arm by which the force is applied. Then, by equation 3, the mechanical advantage $= \frac{2\pi l}{t}$.

Pulleys. If in such an arrangement as that of Fig. 49 the weight were raised one foot, the point of application of F would move six feet; therefore, the mechanical advantage is 6; or, in general, with pulleys arranged as in the figure, the mechanical advantage is equal to the number of cords by which the weight is supported.

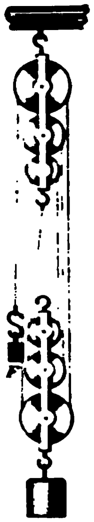


FIG. 49.

Hydraulic Press. The hydraulic press consists of a lever, B (Fig. 50), a pump, H, and a ram, P, working in the cylinder M. Call the distance from the hand to the fulcrum L, and the distance from the piston-rod to the fulcrum l. Then, by the geometrical relations of the figure, the mechanical advantage of the lever will be $\frac{L}{l}$, or, calling F the force applied at B, and F' that exerted on the bottom of the plunger in H,

$$\frac{F'}{F} = \frac{L}{l}. \tag{4}$$

Suppose that the plunger in H moves down a distance, d, and that P moves up a distance, D. Because water is incompressible, the volume displaced in each case

will be the same; or, calling a and A the respective areas of the pistons,

$$DA = dh;$$

whence the mechanical advantage of the pumps is

$$\text{(5)} \qquad \frac{F''}{F'} = \frac{d}{D} = \frac{A}{a}.$$

Multiplying equations 4 and 5, the mechanical advantage of the system is found to be

$$\text{(6)} \qquad \frac{F''}{F} = \frac{AL}{al}.$$

In every actual machine some of the energy is expended in work done against friction or other prejudicial resistances.

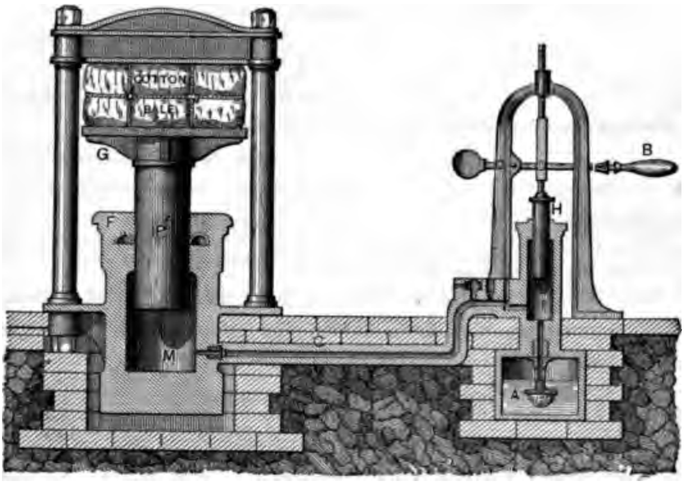


FIG. 50.

The ratio of the work done by the machine to the work done upon it is called the *efficiency*.

52. Friction. — The resistance which bodies oppose to the movement of one surface on another is termed *friction*. It depends both on the nature and the roughness of the

surfaces in contact. At the commencement of the sliding it is greater than when the motion is continued, but does not change much with the relative velocity unless this becomes very great. The ratio of the tangential to the normal stress, when the sliding is about to begin, is called the coefficient of statical friction. The analogous ratio when the motion has become steady is called the coefficient of kinetic friction. Its value varies greatly with the lubrication of the surfaces, and can be determined by experiment only.

EXAMPLES.

1. The lower end of a ladder 15.6 m. long stands on the ground at a distance of 2.73 m. from a wall against which the upper end rests. How much work will be done in carrying 26.4 kilos up the ladder?

Ans. $3.98(10)^{10}$ ergs.

2. The diameter of the cylinder of a steam engine is 18 in. and its length 24 in. What work will be done in each stroke of the piston if the average pressure of the steam is 110 lbs. per sq. in.?

Ans. $0.56(10)^5$ foot-pounds.

3. A mass of 2.64 lbs. attached to a string 39.7 in. long is released after the string has been raised to the horizontal. When the pendulum has fallen to a position making an angle of 30° with the vertical, what will be the kinetic energy of the bob?

Ans. 7.56 foot-pounds.

4. If a mass of 84.5 gms. sliding down a rough inclined plane 78.2 cm. high acquire a velocity of 288 cm. per sec., how much work has been done against friction? *Ans.* $2.97(10)^6$ ergs.

5. 5.92 sec. after a mass of 57.8 gms. has been thrown vertically upward, it is observed to possess a downward velocity of 387 cm. per sec. With what energy will it reach the ground? *Ans.* $0.846(10)^9$ ergs.

6. Given the length of a pendulum 99.3 cm., the maximum displacement 16.3 cm., and the mass of the bob 100 gms., show that the calculated value of the kinetic energy at the middle of the swing is equal to the work done in producing the displacement.

Ans. Kinetic energy $= 1.35(10)^5$ ergs.

7. If a bullet having a velocity of 187 m. per sec. will pierce 8.92 cm. of a target, what velocity must the same bullet have to enter a distance of 14.8 cm.? $v = 241$ m. per sec.

CHAPTER IV.

MECHANICS OF A RIGID BODY.

53. Motion of a Rigid Body. — The treatment of the motion of an extended body may be greatly simplified by assuming that the body is *rigid*, that is to say, suffers no deformation under the action of applied forces.

When a body moves so that the line joining any two points of the body remains parallel to its previous position, the motion is said to be one of *translation*.

When a body moves so that each point describes the arc of a circle having its center on a fixed straight line to which its plane is perpendicular, the motion is said to be one of *rotation*, and the fixed straight line is called the axis of rotation.

The application of one or more forces to a rigid body produces, in general, a change both in the motion of translation and of rotation, which may be treated separately. The first of these changes has already been explained in Chapter II; the second requires further consideration.

54. Moment of a Force. — Suppose that a force, F (Fig. 51), applied to a rigid body at A, produces a rotation of the line OA about the fixed point O, through a small angle θ. Call the distance $OA = p$, and the arc $AB = s$. Taking the work done as the measure of the effect of the force in producing this rotation,

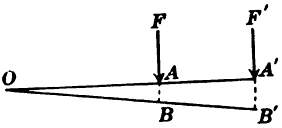


Fig. 51.

(1) $$F \cdot s = Fp\theta.$$

If a different force, F', had been applied at A', its effect would have been

$$F' \cdot s' = F'p'\theta. \tag{2}$$

Whence it appears that, in order that two different forces shall have the same effect in producing a given rotation, it is necessary that

$$Fp = F'p'.$$

This product of the force by the perpendicular on its direction from the center of rotation is called the *moment of the force*, that is to say, the importance of a force in producing a rotation.

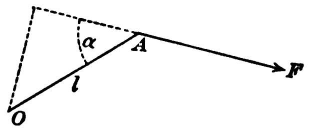


Fig. 52.

If the direction of the force is inclined to OA (Fig. 52) at an angle, α, calling the length of this line l, the moment is

$$F \cdot l \sin \alpha = F \sin \alpha \cdot l,$$

which shows that in discussing moments it is only necessary to consider the component of F, which is at right angles to l.

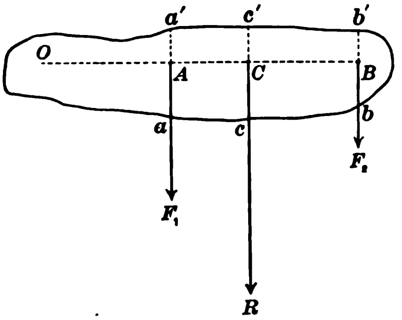


Fig. 53.

55. Resultant of Two Parallel Forces. — Let F_1 and F_2 (Fig. 53) be two parallel forces applied to a rigid body in the direction of the lines aa' and bb'. In order to find the line of application cc' of the resultant, such that the latter shall produce the same effect in rotation about a point, O, as the joint action of F_1 and F_2, draw OB through the

given point perpendicular to the direction of the forces. Let $OA = p_1$, $OB = p_2$, $OC = p$, and call the resultant R. Then in order that the rotating effect of R shall be the same as the combined effect of the forces,

$$Rp = F_1 p_1 + F_2 p_2. \tag{3}$$

But, by definition,

$$R = F_1 + F_2; \tag{4}$$

whence

$$p = \frac{F_1 p_1 + F_2 p_2}{F_1 + F_2}, \tag{5}$$

or,

$$\frac{p_2 - p}{p - p_1} = \frac{F_1}{F_2}. \tag{6}$$

By the figure, $p_2 - p = CB$ and $p - p_1 = AC$; therefore, the point C divides the distance AB into segments which are inversely proportional to the adjacent forces. As the point O may be taken anywhere in the line AB, provided the moments are given the proper sign, say, plus when the rotation would be clockwise and negative in the opposite case, the motion of C at any instant will not be altered if F_1 and F_2 at A and B be replaced by R at C.

Taking O at C, equation 6 becomes

$$\frac{p_2}{p_1} = \frac{F_1}{F_2},$$

or,

$$F_2 p_2 - F_1 p_2 = 0; \tag{7}$$

that is, the moments of F_1 and F_2 about C are equal and opposite.

When F_1 is equal and opposite to F_2, $R = 0$ and the solution fails. This combination of forces illustrated in Fig. 54 is called a *couple*. Taking the moments about O,

$$M = F_2 \cdot \overline{OB} - F_1 \cdot \overline{OA} = F_2 \cdot \overline{AB},$$

or, the moment of a couple is equal to the product of the force into the perpendicular distance between the lines of action of the forces. The value of a couple is, accordingly, independent of its position in the plane.

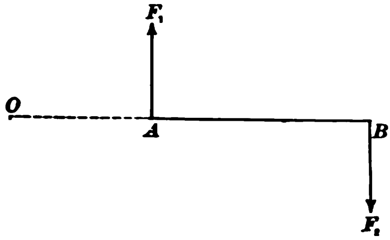


FIG. 54.

56. Center of Parallel Forces. — If any number of parallel forces acting in the same direction be applied to a rigid body, the resultant will pass constantly through a fixed point without reference to the particular direction which the forces may have. To prove this proposition, suppose that forces $F_1, F_2, \cdots F_n$ are applied to the body at points $x_1y_1z_1$, $x_2y_2z_2$, $\cdots$ $x_ny_nz_n$, and let the angle which the common direction of the forces makes with the X axis be called α, that with the Y axis β, and that with the Z axis γ. Denote the resultant of the forces by R; then

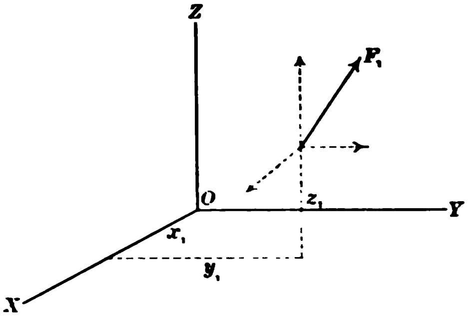


FIG. 55.

$$F_1 + F_2 + \cdots F_n = \Sigma F = R.$$

If R is to be the resultant of $F_1, F_2, \cdots F_n$, *i.e.* completely replace these forces, its moment about any axis must be equivalent to the combined moments of all the individual forces.

Let $\bar{x}, \bar{y}, \bar{z}$, be a point in the line of application of the resultant.

The sum of all the components parallel to the X axis is

(8) $$R \cos \alpha = F_1 \cos \alpha + F_2 \cos \alpha + \cdots F_n \cos \alpha;$$

similarly, for the Y axis,

(9) $$R \cos \beta = F_1 \cos \beta + F_2 \cos \beta + \cdots F_n \cos \beta;$$

and for the Z axis,

(10) $$R \cos \gamma = F_1 \cos \gamma + F_2 \cos \gamma + \cdots F_n \cos \gamma.$$

Now as the moment of R, with respect to any axis, is equal to the sum of the moments of the forces with respect to the same axis, it follows that the moment of the component of R will be equal to the sum of the moments of the components of F_1, F_2, etc., about the same axis. Hence, taking moments about X, Y, and Z,

(11) $$R \cos \alpha \cdot \bar{x} = F_1 \cos \alpha \cdot x_1 + F_2 \cos \alpha \cdot x_2 + \cdots F_n \cos \alpha \cdot x_n,$$

(12) $$R \cos \beta \cdot \bar{y} = F_1 \cos \beta \cdot y_1 + F_2 \cos \beta \cdot y_2 + \cdots F_n \cos \beta \cdot y_n,$$

(13) $$R \cos \gamma \cdot \bar{z} = F_1 \cos \gamma \cdot z_1 + F_2 \cos \gamma \cdot z_2 + \cdots F_n \cos \gamma \cdot z_n,$$

whence

(14) $$\bar{x} = \frac{F_1 x_1 + F_2 x_2 + \cdots F_n x_n}{R} = \frac{\Sigma F x}{\Sigma F},$$

and, similarly,

(15) $$\bar{y} = \frac{\Sigma F x}{\Sigma F},$$

(16) $$\bar{z} = \frac{\Sigma F z}{\Sigma F}.$$

Since the values of $\bar{x}, \bar{y}, \bar{z}$, are independent of direction of the forces, it follows that if the system of forces, $F_1, F_2, \cdots F_n$, be applied in any direction, at the same points, the

resultant would always pass through a single point. This point has received the name of center of parallel forces.

57. Center of Gravity. — If m_1, m_2, $\cdots$ (Fig. 56) be a system of particles rigidly connected, the weights of these particles may be regarded as forces having sensibly the same direction. The point G, through which the resultant of this system of parallel forces constantly passes when the system is turned into different positions, is called the *center of gravity*. Substituting the weights of the elementary masses for the forces in equations 14, 15, and 16, the coördinates of the center of gravity are found to be

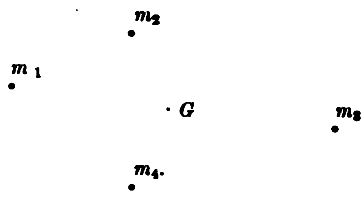


FIG. 56.

$$
(17) \qquad \begin{cases} \bar{x} = \dfrac{\Sigma mgx}{\Sigma mg} = \dfrac{\Sigma mx}{\Sigma m} \\[2ex] \bar{y} = \dfrac{\Sigma mgy}{\Sigma mg} = \dfrac{\Sigma my}{\Sigma m} \\[2ex] \bar{z} = \dfrac{\Sigma mgz}{\Sigma mg} = \dfrac{\Sigma mz}{\Sigma m}. \end{cases}
$$

The point defined by the final form of these equations, though identical with the center of gravity, might be more properly called the center of mass, since weight does not enter into its definition. The solution of these equations in general requires the aid of more powerful mathematical methods than may profitably be introduced here, but in the simplest cases, which are also the most useful, the position of the center of gravity may be found by the application of elementary considerations alone.

The Center of Gravity of a Straight Line. Since the resultant of two equal parallel forces bisects the line joining

them (Art. 55), the center of gravity of two equal particles is half-way between the particles. Now, as any uniform physical line may be regarded as made up of pairs of such particles, the center of gravity of a line will be at its middle point.

Center of Gravity of a Lamina having an Axis of Symmetry. Any indefinitely thin lamina may be regarded as made up of a number of parallel lines. If it has an axis of symmetry, these lines may be chosen so that they will be bisected by this axis. Now, as the center of gravity of each of the lines is on the axis, the center of gravity of the lamina will also lie on the axis.

It follows that, if the lamina have two axes of symmetry, the center of gravity will be at the intersection. Therefore, the center of gravity of any regular plane figure is at the geometrical center.

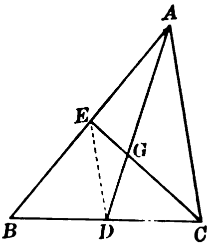


FIG. 57.

Center of Gravity of a Triangle. The center of gravity of a triangle, G (Fig. 57), is at the intersection of two median lines AD and CE. Because AB and BC are bisected at E and D, ED is parallel to AC, and

$$\frac{DE}{AC}=\frac{BD}{BC}=\frac{1}{2}.$$

Also, since the triangles AGC and DGE are similar,

$$\frac{DG}{AG}=\frac{DE}{AC}=\frac{1}{2},$$

or,

$$DG \text{ is } \tfrac{1}{3}\, AD.$$

The center of gravity of any polygon may be found by dividing it into triangles, and regarding the weight of each of these applied at their respective centers of gravity.

Center of Gravity of a Solid having a Plane of Symmetry. If a solid have a plane of symmetry, it may be regarded as made up of laminae which are arranged in equal pairs with respect to this plane. As the center of gravity of each pair of equal laminae is in the plane, the center of gravity of the whole solid will lie there. If there are two planes of symmetry, it will lie on the line of intersection, and if three, at the point common to all. Hence the center of gravity of a regular solid is at its geometrical center.

Center of Gravity of a Tetrahedron. Consider the tetrahedron $ABCD$ (Fig. 58) as made up of laminae which are similar to the base BCD. Then the centers of gravity of each of these will lie on the line AE, drawn from the vertex to the center of gravity of the base BCD, and hence the center of gravity of the tetrahedron will be found somewhere in this line. By similar reasoning it may be shown to lie in the line BF, drawn from B to F, the center of gravity of ADC. Now, as BF and AE are both in the plane AHB, they must intersect in some point, G, whose position is required. By the construction of the figure, $FH = \frac{1}{3}\, AH$ and $EH = \frac{1}{3}\, BH$; therefore, EF is parallel to AB, and the triangles AGB and GEF are similar.

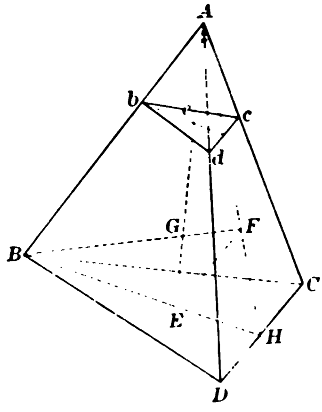


FIG. 58.

Whence

$$\frac{EG}{AG} = \frac{EF}{AB};$$

but

$$\frac{EF}{AB}=\frac{HF}{HA}=\tfrac{1}{3},$$

therefore,

$$\frac{EG}{AG}=\tfrac{1}{3}, \text{ or } EG=\tfrac{1}{4}\,EA.$$

Since every pyramid may be divided into a number of tetrahedra having a common vertex, and their bases in the base of the pyramid, the center of gravity of a pyramid will lie in a plane drawn parallel to the base at one-fourth the altitude above it. Also, as the pyramid may be considered as made up of laminae which are polygons parallel to the base, the center of gravity of the pyramid must lie on the line joining the vertex to the center of gravity of the base, and one-fourth the distance, as was just shown.

The center of gravity of a cone is likewise seen to lie on the axis, at one-fourth the distance to the vertex, since any cone may be regarded as a pyramid of an infinite number of sides.

Center of Gravity of a Compound Figure. When a body is made up of several parts, of which the centers of gravity are known, the center of gravity of the whole may readily be found by application of equations 17. For instance, let it be required to find the center of gravity of a trapezoid. This quadrilateral (Fig. 59) may be regarded as composed of a triangle and a parallelogram.

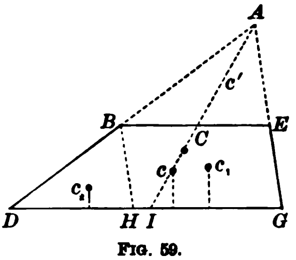


Fig. 59.

Call the mass of the triangle BDH, m_1, the mass of the parallelogram $BEGH$, m_2, and the mass of the trapezoid

$DBEG$, $m = m_1 + m_2$. Let u be the distance of c from the base DG; then

$$\bar{u} = \frac{m_1 u_1 + m_2 u_2}{m_1 + m_2}.$$

Calling $BE = a$, $DG = b$, and the altitude $= h$, and assuming that the density and the thickness of the body are constant, the value of $\bar{u}$ reduces to

$$\bar{u} = \tfrac{1}{3} h \frac{(b + 2a)}{(b + a)}.$$

It is also evident that the trapezoid might be regarded as a portion of the triangle ADG, in which the center of gravity C of the whole, and c' of the part ABE are known. Calling distances measured on the median from I, v, the mass of $ADG = M$ and $ABE = m'$;

$$v = \frac{M\bar{v} - m'v'}{m} = \tfrac{1}{3} s \frac{b + 2a}{b + a},$$

where s is the portion of the median included between BE and DG.

58. Conditions of Equilibrium. — If the sum of all the moments of the forces applied to a rigid body is zero, the forces will produce no change of rotation. If, in addition, the sum of the forces is zero, they will produce no change of translation. Therefore, the conditions for equilibrium may be written

$$\begin{cases} \Sigma Fp = 0, & (18) \\ \Sigma F = 0, & (19) \end{cases}$$

provided the moments and the forces in each case are taken with the proper algebraic sign. The origin of moments is immaterial, since every point is at rest. As a simple illustration of the preceding principles, suppose a bar 5 ft. long, weighing 30 lbs., has its center of gravity 3 ft. from one

end; if 10 lbs. be hung on this end and 25 lbs. on the other, required the point at which it will balance, and the weight sustained by the support.

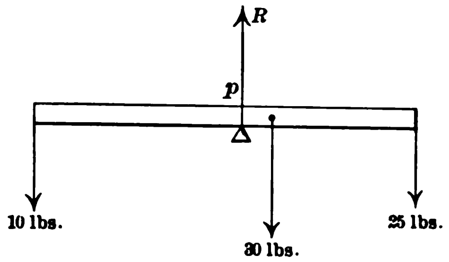


FIG. 60.

Let any point in the diagram (Fig. 60) be taken as the axis of moments, as, for example, the center. Call R the reaction of the support placed at a distance, p, from the center. Calling the downward rotation of the right-hand end positive, equations 18 and 19 here become

$$(25 \text{ lbs.} \times 2\tfrac{1}{2} \text{ ft.} + 30 \text{ lbs.} \times \tfrac{1}{2} \text{ ft.} - 10 \text{ lbs.} \times 2\tfrac{1}{2} \text{ ft.})\, g - Rp = 0,$$
$$(25 \text{ lbs.} + 30 \text{ lbs.} + 10 \text{ lbs.})\, g - R = 0,$$

whence $R = 65$ lbs. g, and $p = 0.81$ ft.

59. Motion of Rotation of a Rigid Body.—Suppose that F_1, F_2, and F_3 are a series of forces applied to separate particles, m_1, m_2, m_3 (Fig. 61), originally in a straight line, so that

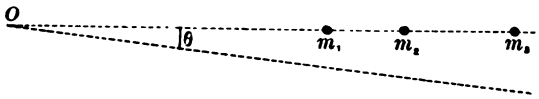


FIG. 61.

at the end of a short time they have each changed their angular position, with respect to the point O, by an amount θ. Also let r_1, r_2, r_3 denote the distance of each particle from O, and s_1, s_2, s_3 denote the linear distances traversed by the corresponding particles. It will be convenient to use here a notation in which the time rate of change of a

quantity is denoted by a dot written over the symbol of the quantity. Thus, velocity will be written

$$\dot{s} = \underset{t=o}{\text{limit}} \frac{s_t - s_o}{t}.$$

Similarly, acceleration will be denoted by

$$\ddot{s} = \underset{t=o}{\text{limit}} \frac{\dot{s}_t - \dot{s}_o}{t},$$

and so on.

The forces in Fig. 61 may now be written

$$(20) \qquad \begin{cases} F_1 = m_1 \ddot{s}_1, \\ F_2 = m_2 \ddot{s}_2, \\ F_3 = m_3 \ddot{s}_3, \end{cases}$$

or,

$$(21) \qquad \begin{cases} F_1 = m_1 r_1 \ddot{\theta}_1, \\ F_2 = m_2 r_2 \ddot{\theta}_2, \\ F_3 = m_3 r_3 \ddot{\theta}_3, \end{cases}$$

since $\ddot{s} = \ddot{\overline{r\theta}} = r\ddot{\theta}$, for r is constant.

Multiplying each of the equations 21 by the distance of the particle from the axis,

$$(22) \qquad \begin{cases} F_1 r_1 = \ddot{\theta} m_1 r_1^2, \\ F_2 r_2 = \ddot{\theta} m_2 r_2^2, \\ F_3 r_3 = \ddot{\theta} m_3 r_3^2; \end{cases}$$

whence, by addition,

$$(23) \qquad \Sigma Fr = \ddot{\theta} \Sigma mr^2.$$

Since the particles in Fig. 61 remain in the same relative positions with respect to each other, nothing would be altered by supposing them rigidly connected. That is to say, equation 23 may be applied to any rigid body rotating about a fixed axis. Its meaning expressed in words is that

the moment of the forces applied to a body is equal to the product of the angular acceleration by a certain function which may be found by taking the sum of the products of each of the elementary masses by the square of its distance from the axis of rotation. This function is called the *moment of inertia.*

Since $m\dot{s}$, the product of mass and velocity, is called momentum, $m_1\dot{s}_1r_1 = m_1r_1^2\dot{\theta}$ has received the name *moment of momentum.*

Taking the sum of all such quantities for a rigid body,

$$\Sigma m\dot{s}r = \dot{\theta}\Sigma mr^2, \tag{24}$$

or, the moment of momentum is equal to the angular velocity multiplied by the moment of inertia. Taking the time rate of change of this quantity,

$$\dot{\overline{m_1\dot{s}_1r_1}} = \dot{\overline{m_1r_1^2\dot{\theta}}} = \ddot{\theta}m_1r_1^2 = F_1r_1; \tag{25}$$

that is to say, the moment of the forces applied to a rigid body is measured by the time rate of change of the moment of momentum.

Again, since the kinetic energy of any elementary mass revolving about O (Fig. 61) is

$$\tfrac{1}{2}m_1\dot{s}^2 = \tfrac{1}{2}\dot{\theta}^2m_1r_1^2,$$

taking the sum of the energy of all the parts of the body,

$$E = \Sigma\tfrac{1}{2}m\dot{s}^2 = \tfrac{1}{2}\dot{\theta}^2\Sigma mr^2; \tag{26}$$

that is to say, the kinetic energy of a rotating body is found by multiplying one-half of the square of the angular velocity by the moment of inertia.

Reviewing these results, it appears that for every equation of motion with simple translation, *e.g.*

$$F = mf,$$
$$E = \tfrac{1}{2}mv^2,$$

there exists an analogous one in rotation, such as

$$M = (\Sigma mr^2)\ddot{\theta},$$
$$E = \tfrac{1}{2}(\Sigma mr^2)\dot{\theta}^2,$$

in which the moment of the forces replaces the force, the moment of inertia replaces the mass, angular velocity replaces linear velocity, and angular acceleration replaces linear acceleration. The significance of the term moment of inertia is, the importance which mass (inertia) plays in problems of rotation.

60. Radius of Gyration. — Just as in the case of center of gravity it is possible to find, by the equation

$$\bar{x} = \frac{\Sigma mx}{\Sigma m},$$

some point in the body such that the moment of the resultant force applied at this point shall be equal to the sum of the moments of the distributed forces, or, in other words, some point at which all the mass may be regarded as concentrated for problems of equilibrium or of motion of translation, so it is possible to find a distance from the axis of rotation such that the energy of rotation, the angular velocity, and the angular acceleration of an extended body shall be the same as if the total mass were concentrated at this distance from the axis of rotation. The name given to this distance is the *radius of gyration*. It is defined by k in the following equation:

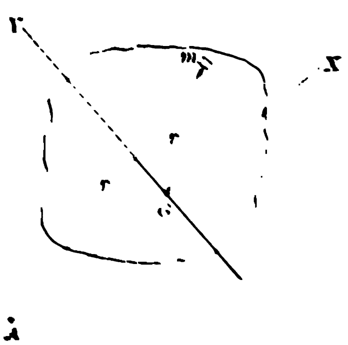


Fig. [illegible]

$$(27) \qquad (\Sigma m)k^2 = \Sigma mr^2.$$

The calculation of k, even for simple cases, is not readily effected without the use of the Integral Calculus[1] and hence need not be discussed here. However, in further illustration of the principles laid down, the following theorem will be proved and applied to the case of the physical pendulum.

[1] See Williamson's Integral Calculus, Sixth Edition, p. 291.

61. Theorem. — The moment of inertia about any axis is equal to the moment of inertia about a parallel axis through the center of gravity increased by the product of the mass of the body by the square of the distance between the parallel axes.

Let G be the center of gravity of the body and A the position of the parallel axis.

Then will

$$\Sigma mr^2 = \Sigma mr^2 + \Sigma ma^2.$$

Choose the line passing through A and G as the axis of X; then

$$\Sigma mr^2 = \Sigma m(x^2 + y^2), \tag{28}$$

and

$$\begin{aligned} \Sigma mr'^2 &= \Sigma m[(x + a)^2 + y^2] \\ &= \Sigma m[x^2 + y^2 + 2ax + a^2] \\ &= \Sigma m(x^2 + y^2) + 2\Sigma max + \Sigma ma^2; \end{aligned}$$

but $\quad \Sigma ma^2 = a^2\Sigma m,$

and, by equation 17,

$$\Sigma mx = \bar{x}\Sigma m = 0,$$

for the center of gravity;

$$\therefore \Sigma max = a\Sigma mx = 0;$$

and, finally,

$$\Sigma mr'^2 = \Sigma mr^2 + a^2\Sigma m, \tag{29}$$

which becomes, on substitution of the radii of gyration,

$$k'^2 = k^2 + a^2. \tag{30}$$

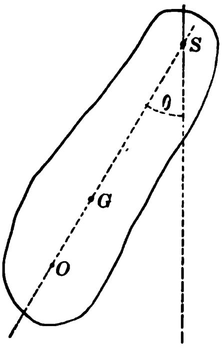


FIG. 63.

62. Compound Pendulum. — Let a physical pendulum be suspended from the point S (Fig. 63), and suppose its center of gravity is at G. Denote SG by a, and the angle made by this line with the vertical at any instant by θ. Let the moment of inertia about S be

$$\Sigma mr_1^2 = k_1^2\Sigma m,$$

and the maximum displacement of SG from the vertical α.

When the pendulum has fallen through an angle, $\alpha - \theta$, and G lowered by a distance, h, the work done by the force of weight will be

$$gh\Sigma m = ga(\cos\alpha - \cos\theta)\Sigma m,$$

and the kinetic energy acquired

$$E_k = \tfrac{1}{2}k_1{}^2\dot{\theta}^2\Sigma m.$$

Since this is a conservative system,

$$ga(\cos\alpha - \cos\theta)\Sigma m = \tfrac{1}{2}k_1{}^2\dot{\theta}^2\Sigma m. \tag{31}$$

Next consider a simple pendulum of length l and mass Σm which has fallen through the same angle $\alpha - \theta$. The work done on this pendulum will be $\Sigma mgl(\cos\alpha - \cos\theta)$, and the kinetic energy acquired

$$\tfrac{1}{2}v^2\Sigma m = \tfrac{1}{2}l^2\dot{\theta}_1{}^2\Sigma m,$$

if $\dot{\theta}_1$ denote the angular velocity of the simple pendulum.

Whence

$$gl(\cos\alpha - \cos\theta)\Sigma m = \tfrac{1}{2}l^2\dot{\theta}_1{}^2\Sigma m. \tag{32}$$

Equating the value of $\dot{\theta}$ from equation 31 with $\dot{\theta}_1$ from equation 32, we find the condition that the simple and compound pendulums shall have acquired the same angular velocity.

$$l = \frac{k_1{}^2}{a} = \frac{k^2 + a^2}{a} \text{ by equation 30.} \tag{33}$$

Since this expression is independent of the displacement of the pendulum, it follows that the compound pendulum will swing in the same time as a simple one whose length is $\dfrac{k_1{}^2}{a}$.

Hence the period of a compound pendulum by equation 65 (Art. 40) is

$$P = 2\pi\sqrt{\frac{k_1{}^2}{ag}}$$

$$= 2\pi\sqrt{\frac{k^2 + a^2}{ag}} = 2\pi\sqrt{\frac{\Sigma m r_1{}^2}{ga\Sigma m}}. \tag{34}$$

63. Reversible Pendulum. — Let the point O (Fig. 63) be chosen so that $GO = \dfrac{k^2}{a}$; then, by equation 33,

$$SO = SG + GO = a + \frac{k^2}{a}$$

is the length of the equivalent simple pendulum.

If the physical pendulum be reversed so that O is made the center of suspension, the length of the equivalent simple pendulum will be

$$OG + \frac{k^2}{OG} = \frac{k^2}{a} + \frac{k^2}{\frac{k^2}{a}} = \frac{k^2}{a} + a.$$

In other words, the pendulum will swing in the same time whether suspended from S or O. The point O is called the *center of oscillation.*

The application of the reversible pendulum to the determination of g has already been discussed in Art. 42.

EXAMPLES.

1. A weight of 390 pounds is carried by a bar, AB, 13.2 ft. long, supported at the ends. If the distance of the weight from the end A is 5.6 ft., what is the weight borne by each support?

$A = 225$ pounds.
$B = 165$ "

2. A uniform lever 9.6 ft. long weighs 12.3 pounds. If 24.5 pounds are hung at one extremity and the bar is supported 2.84 ft. from this end, what weight must be applied at the opposite end in order that the system shall be in equilibrium? *Ans.* 6.73 pounds.

3. A lever, AB, 7.56 ft. long, weighing 17.3 pounds, has its center of gravity at a distance of 2.37 ft. from A. What weight at A will be necessary to sustain 3.54 pounds at B, when the bar is supported at a distance of 1.22 ft. from the end A? *Ans.* 34.7 pounds.

4. A lever, AB, 6.34 ft. long balances about a point 2.42 ft. from A. When loaded with 35.5 pounds at A and 5.78 pounds at B, it balances at a point 1.07 ft. from A. What is the weight of the lever? *Ans.* 5.56 pounds.

5. It is found that a lever cut from a bar weighing 4.2 pounds to the foot balances at a distance of 2.3 ft. from one end when weighted at this end with 54 pounds. What is the length of the bar? *Ans.* 10.3 ft.

6. When a lever, AB, is supported at its center of gravity, it is found that a weight, W, hung at A will balance 2.5 pounds at B; but when W is hung at B it requires a weight of 19 pounds at A to keep it in equilibrium. What is the weight W? *Ans.* 6.9 pounds.

7. Find the magnitude and line of application of the resultant of two oppositely directed forces, equal, respectively, to 25 and 42, when their lines of action are 3.4 ft. apart.

Ans. 17; 5 ft. beyond the larger force.

8. The beam of a steelyard weighs 1.2 pounds; the distance of the center of gravity from the fulcrum is 0.27 in. toward the shorter end; the distance from the fulcrum to the point of support of the body to be weighed is 0.78 in.; the counterpoise weighs 0.61 pound. Where is the zero of the scale, and what is the length of the graduation for 1 pound?

Zero = 0.53 in. from fulcrum; divisions = 1.28 in. long.

9. If a mass weigh 29.62 gms. in one pan of a balance and 28.71 gms. in the other, what is its true weight? *Ans.* 29.16 gms.

10. A uniform rod 24 in. long is bent to an angle of 90° at its middle point. What is the position of the center of gravity?

Ans. 8.5 in. from the angle.

11. A board originally square, with sides 24 in. long, has a corner cut off by a line joining the middle points of two adjacent sides. What is the distance of the center of gravity from the center of the square?

Ans. 1.6 in.

12. A uniform wire is bent into the form of a triangle, each of whose sides is 117 cm., and whose base is 90 cm. Where is the center of gravity?

Ans. 39 cm. from the base on the perpendicular to its middle point.

13. Find the center of gravity of a quadrilateral made by joining the bases of two isosceles triangles whose altitudes are, respectively, 61 cm. and 118 cm.

Ans. 80 cm. from the apex of the smaller triangle.

14. A sphere 268 cm. in diameter contains a spherical cavity whose center is 59 cm. from the center of the sphere and whose diameter is 134 cm. Where is the center of gravity of the body?

Ans. 8.43 cm. from the center of the sphere.

15. A uniform bar is supported by a smooth peg so that one end rests against a wall. If the distance of the peg from the wall is $\frac{1}{n}$th of the length of the bar, what angle will the bar make with the horizontal when in equilibrium? *Ans.* $\theta = \cos^{-1}\left(\frac{2}{n}\right)^{\frac{1}{3}}$.

16. If masses m_1, m_2, at hights h_1 and h_2 above the ground, be raised through distances s_1 and s_2, respectively, show that the work done will be equal to that required to elevate the total mass through the distance which the center of gravity is raised.

17. A vertical U-tube having a uniform bore of 2.56 cm. radius is partly filled with mercury. How much work will be done in producing a difference of level in the two legs of 12.3 cm.?

Ans. $1.04(10)^7$ ergs.

18. A long chain having a length, l, and a mass, m, is attached by one end to a drum. How much work will be done in winding up two-thirds of the chain? *Ans.* $\frac{4}{9}$ mgl.

NOTE. Consider the rise of the center of gravity of the chain.

CHAPTER V.

ELASTICITY.

64. Stress. — The mutual action of one body on another, or of one part of a body upon another part, is called a *stress*.

In a complete discussion of the phenomena of forces, every force must be regarded as acting between two bodies or systems of bodies. When a force is considered as acting across a certain area, the measure of the stress at any point is taken as the force per unit area at this point. The dimensions of stress are $ML^{-1}T^{-2}$, and the unit a stress of one dyne per square centimeter. This unit is symbolized by $\frac{1 \text{ gm.}}{1 \text{ cm. } 1 \text{ sec.}^2}$, but has received no accepted name.

When the effect of a stress at a section of a body is such as to increase the dimension of the body at right angles to this section, the stress is called a *tension*. When the effect is to diminish this dimension, it is called a *pressure*. A tension in one direction, combined with an equal pressure at right angles, is called a *shearing stress*. If, for instance, the cube $ABCD$ (Fig. 64) be subjected to a pressure perpendicular to the sides AB and CD, and a tension perpendicular to AD and BC, the resulting stress will be tangential at sections parallel to DB or AC, and have the effect of sliding one portion of the body on another, in these directions, as when cut by a pair of shears.

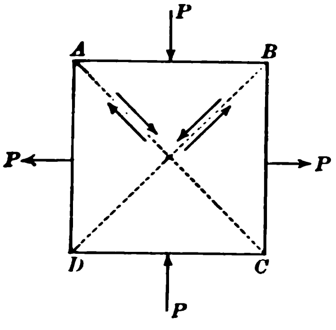


FIG. 64.

When the pressure at a point is the same in all directions, the stress is called a *hydrostatic pressure*.

On account of the convenience with which the pressure of a gas may be measured by the column of mercury which it will support, the pressure at the depth of one centimeter in a pool of mercury is often used as a unit instead of the dyne per square centimeter. Since the density of mercury is 13.6 gms. per cc., and the value of g is 980 $\frac{\text{cm.}}{\text{sec.}^2}$, this new unit, commonly called a "centimeter of mercury," is equal to $1.33(10)^4$ dynes per square centimeter.

The gravitational unit of force (Art. 13) leads to gravitational units of pressure, also, such as the gram force per square centimeter, or the pound per square inch, which are occasionally used.

65. Strain. — The deformation resulting from a stress is called a *strain*. It is measured by the ratio of the change of a dimension to its unstrained value, and is therefore a pure number.

66. Elasticity. — Elasticity is the property by virtue of which a body recovers from a strain. If a body under stress experiences a strain of definite amount, which does not change when the time is prolonged and which disappears completely when the stress is removed, it is said to be *perfectly elastic*. Probably a majority of bodies are perfectly elastic under hydrostatic pressure, but no substance yet tried is perfectly elastic under shearing stress, except perhaps for very small values of the stress.

If the form of a body is permanently altered when the stress exceeds a certain amount, the body is said to be *plastic*. When the alteration is about to take place, the body is said to have reached the *limit of elasticity*.

The *strength* of a body is the value of the stress when it breaks or gives way altogether.

If a body breaks when the strain exceeds a certain small amount, it is called *brittle*. Glass, porcelain, and gems are examples of brittle bodies. Plastic bodies which may be beaten out into thin leaves are termed *malleable*. Gold is a very malleable substance. Those bodies which may be drawn out into fine wires are called *ductile*. Silver, copper, and iron are familiar examples.

If a constant stress produces a strain which increases continually with the time, the substance is called *viscous*.

67. Forms of Matter. — All bodies may be divided into two classes, solids and fluids, according to their behavior under stress.

A *solid* is a body which offers resistance to both change of volume and change of shape, or, more briefly, supports a shearing stress.

A *fluid* is a body which cannot remain in permanent equilibrium under a stress which is different in different directions.

There is no well-marked division between these classes. In reality one shades off gradually into the other.

When a continual yielding is produced only by forces exceeding a certain value, the body must be regarded as solid, however soft it may be.

By dissolving gelatine in water, it is possible to form a series of compounds passing from the hard gelatine, through the weak but solid jellies to the certainly fluid water.

On the other hand, substances which are hard and brittle under large stresses may yield continually to the smallest forces if [illegible] are applied long enough. A stick of sealing wax is an example of such a body. [illegible] be laid horizontally

between two supports, it will be found to bend continually under its own weight. Sealing wax must, therefore, be regarded as a very viscous fluid. Pitch exhibits similar properties, as has been shown by the following experiment due to Lord Kelvin. A thick cake of shoemaker's wax was laid on some corks, and a number of bullets placed on the upper surface of the cake, the whole being immersed in a large mass of water to prevent sudden temperature changes. After a few months the corks were found to have worked their way to the top, and the bullets to have penetrated to the bottom.

By a combination of pitch with tar or turpentine, it is possible to construct a series of compounds of varying viscosity, from an apparently brittle solid to a very mobile fluid.

68. Fluids. — Fluids may be divided into two distinct types, liquids and aeriform bodies.

A *liquid* is a body which offers resistance to increase or diminution of its volume.

An *aeriform* body is one which offers resistance to diminution, but none to the increase of its volume.

That liquids resist stresses which act to increase their volume is not a very familiar fact, because of the ease with which they flow. Water, the commonest example of a liquid, may be shown to possess considerable strength by the apparatus represented in Fig. 65.

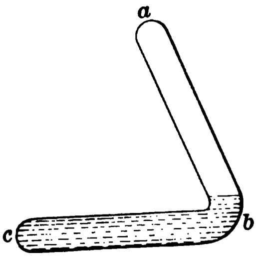


Fig. 65.

abc is a bent tube of glass partly filled with water, and sealed while the water was boiling, so that the space

above the liquid contains a small amount of steam free from air.

If the tube has remained for some time in such a position as to keep the leg *cb* full, it may be carefully reversed, as in Fig. 66, without the water flowing out of the leg *cb*. The stress at any section, *d*, is produced by the weight of the column *de* diminished by the pressure of the steam in the leg *ab*. If the hight of the section considered, at ordinary temperatures, be more than 25 cm., the liquid will be under tension at that point, and it is sometimes possible to strike the end *b* a considerable blow before the column of water will break. Water under these circumstances has been known to support a tension of 800 lbs. per square inch.

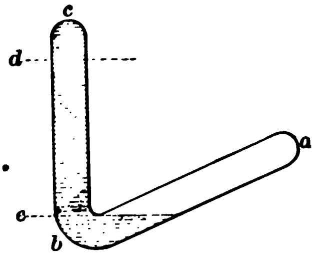


FIG. 66.

Since liquids do not change their volume except under the action of applied forces, it is possible to keep a vessel partly full of a liquid. In such a case the liquid will have one or more free surfaces, *i.e.* surfaces separating it from an aeriform body.

On account of the perfect miscibility of all substances in the aeriform condition, bodies in this state cannot have a free surface.

Another convenient distinction between liquid and aeriform fluids, arising from the possibility of a free surface, is the fact that a liquid may be poured in drops, but an aeriform body cannot.

69. Aeriform Bodies. — Aeriform bodies are conveniently divided into two classes, gases and vapors.

A *gas* is an aeriform fluid in which the product of the pres-

sure by the volume is constant, if the temperature remain unchanged. Air, hydrogen, and oxygen are examples of gases.

A *vapor* is an aeriform fluid in which at a constant temperature this product is not constant. Steam in contact with water is an example of vapor.

Probably all substances are capable of existing in each of the physical states, under proper conditions of volume, pressure, and temperature. Further discussion of the properties of aeriform bodies, as well as the change from one state to another, will be found in the chapters on Heat.

70. Structure. — The great majority of substances when they solidify show a marked tendency to assume symmetrical forms bounded by plane faces. Such bodies are called *crystalline*. Familiar examples are ice, sugar, sulphur, and most gems.

Those substances which exhibit no such tendency are termed *amorphous*. Glass, gums, and starch are examples of this class.

Most crystalline solids show differences in their physical properties according to the direction along which they are tested. For example, the mineral selenite ($CaSO_4$) may be split in one direction into thin laminae with vitreous or glass-like surfaces; in another direction it breaks less readily and shows a conchoidal or shell-like surface; in a third definite direction the broken surface has a fibrous appearance.

Calcite ($CaCO_3$) is another mineral with three cleavage planes, but splitting with equal ease along each. When heated, this mineral expands in a certain direction, but contracts in all directions at right angles. It also transmits light with different velocities in the directions mentioned.

When the structure of a body is such that one small part

is just like every other part, the body is said to be *homogeneous*. Crystallized solids and metals, such as silver, when melted and allowed to cool slowly, are believed to be very nearly homogeneous.

Bodies which have not this property are called *heterogeneous*. They are exemplified by cork, granite, shells, etc.

71. Crystals. — A portion of a crystalline substance formed by continuous and spontaneous growth is called a *crystal*.

The study of the geometric forms which substances assume belongs to crystallography and need not be discussed here. It may, however, be remarked that throughout considerable variation in size and regularity of crystal formation, the angles between the bounding faces in each substance occur with a constancy so characteristic as to furnish a valuable means of identification.

The most remarkable fact which has been discovered in the study of crystals may be stated as follows: if the faces of a crystal be given by their intercepts on axes that meet but do not lie in the same plane, the ratios of these quantities can be rigorously expressed by small whole numbers. The physical interpretation of this law appears to be that the crystal is built up of little particles or *molecules* systematically arranged.

The hypothesis of the molecular constitution of bodies is abundantly justified by the assistance it renders in the explanation of phenomena in all branches of Physics.

The present state of the theory may be briefly sketched as follows.

72. Molecular Theory of the Constitution of Bodies. — All bodies consist of a finite number of parts called molecules.

Each molecule may consist of several distinct kinds of

matter, so held together by chemical bonds that the whole may be set rotating, vibrating, or moving in any way as a single system.

All the molecules of the same substance are alike in composition, grouping, and quantity of matter.

The molecules of a body are in a continual state of agitation, which grows more violent the hotter the body becomes.

The path which the molecule of a solid may describe is restricted to a very small region of space.

In the fluid state the molecule is free from such restriction, though in general it can travel but a small distance before it is disturbed by an encounter with another molecule. After the encounter there is nothing to determine whether it shall return to its original position or push out into some other region.

The mean free path of a molecule in the aeriform condition is much greater than in the liquid state.

The smallest portion of each kind of matter entering into composition is called an *atom*. At present, nearly ninety different elements, or kinds of matter, are known.

73. Temper. — The form assumed by a body where each particle is free to choose the position of least potential energy with respect to those already deposited, as is the case in the slow growth of a crystal from a solution or vapor, is essentially one of stable equilibrium. The case is very different when a body is suddenly cooled. The molecules may then be so arranged that the parts of the body are under varying stress.

This state is well exemplified by Prince Rupert's drops, which are formed by allowing drops of melted glass to fall into water. On snipping off the tail, the whole mass falls to powder with an explosion.

The hardening of steel by plunging it, when red hot, into a cold bath, as well as the hardening of metals by rolling or drawing, is due to a particular arrangement of the molecules. The internal stresses are relieved by subjecting the body to a gentle heat for some time.

The latter process is called *tempering*, when the softening is slight, and *annealing*, when it is as great as possible.

74. Coefficients of Elasticity. — If any stress be denoted by P, and the corresponding strain by p, then, if the strain is small, the results of experiment may be accurately expressed by the relation

$$(1) \qquad P = cp,$$

where c is a physical quantity called the coefficient of elasticity. That is to say, the coefficient of elasticity is defined as the quotient of stress by strain.

The three kinds of elasticity most commonly studied are: 1°, elasticity of volume; 2°, the longitudinal elasticity of a body free to expand or contract laterally; and 3°, shearing elasticity.

75. Elasticity of Volume. — Suppose a body is subjected to a uniform normal stress, P, over its whole surface.

If V denote the original volume, and $V - v$ the strained volume, $\frac{v}{V}$ is taken as the measure of the strain.

Let k denote the coefficient of elasticity of volume; then, by equation 1,

$$(2) \qquad k = \frac{PV}{v}.$$

By definition this is the only elasticity possessed by liquids and gases. The reciprocal of k is called the compressibility.

76. Young's Modulus. — Let L be the unstrained length of a body free to expand or contract laterally, and $L \pm l$ the strained length. Then, taking $\frac{l}{L}$ as the measure of the strain produced by a force, F, applied longitudinally to a rod of cross section, A, the quotient $\frac{\text{stress}}{\text{strain}}$ is known as Young's Modulus.

Denoting it by M,

(3) $$M = \frac{FL}{Al}.$$

77. Shearing Elasticity. — The elasticity exhibited by a solid when distorted so as to leave the volume unchanged is called shearing elasticity or simple rigidity.

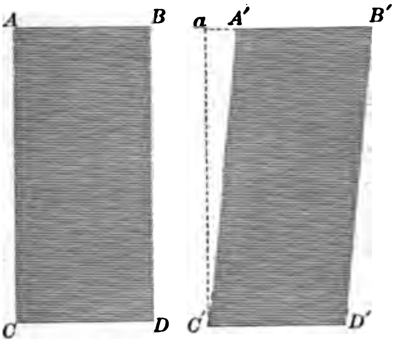


FIG. 67.

In illustration of such a strain, called a *shear*, consider Fig. 67 to represent a pile of papers which has been deformed by sliding each sheet upon the one below it a small amount. Since the area of the base of the pile and the hight remain unaltered, the volume has not been changed. The angle between AC and $A'C'$ is taken as the measure of the strain; thus, the

$$\text{shear} = \frac{\overline{aA'}}{\overline{aC'}}.$$

The coefficient of shearing elasticity or simple rigidity is defined by

$$n = \frac{\text{shearing stress}}{\text{shearing strain}}.$$

78. Torsion. — Imagine a cylinder (Fig. 68) to be made up of a series of transverse slices. The effect of a twisting couple will be to slide each slice on the one preceding; the strain will therefore be a shear. If ab be an element of the cylinder in the unstrained, and ac the position of the same line in the strained condition, the measure of the shear will be for the outer layer $\frac{bc}{ac}$. Let fh (Fig. 69) be an elementary area, at a distance, r, from the axis, and situated in a slice at a distance, $ab = l$, from the upper end. Call θ the angular displacement of the slice. Then the shear for this element is measured by $\frac{r\theta}{l}\cdot$ Since

FIG. 68.

the stress is proportional to the shear, and the moment of the stress varies as the stress, it follows that the moment of restitution of the stress varies as $\frac{\theta}{l}$. But the moment for any other element in the slice has the same value of θ and l. Therefore, the moment of restitution for the entire section varies directly as the angular displacement, and inversely as the length; or, in symbols,

$$M_T \propto \frac{\theta}{l}\cdot$$

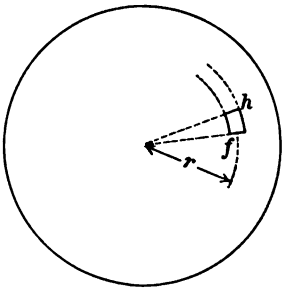


FIG. 69.

Comparing the first of these laws with $\Sigma F \cdot r = \frac{4\pi^2}{P}\theta(\Sigma mr^2)$, it follows that torsional vibrations should be harmonic.

It remains to show how the moment depends on the radius. Consider two rods, A, B, of the same length (Fig. 70), the radii of which are in the ratio of 1 to 2, and let each be

twisted through an angle, θ. Further suppose that each section is divided up into the same number of concentric rings. Fixing the attention on the corresponding ring in each section, it appears that in B the area is four times as great. Therefore, the force of restitution, which is stress $\times$ area, is four times as great on account of the area.

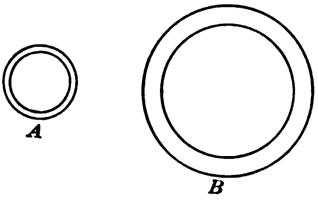


FIG. 70.

Also, the shear is twice as great in B as in A; therefore, the force is twice as great on account of the shear. But the lever arm of the force in B is twice as great as in A; therefore, the moment of restitution is twice as great on account of the distance from the center. Combining these different values, it appears that the moment of restitution in B is $4 \times 2 \times 2$, or 2^4 times that in A.

Wherefore, this moment varies as the fourth power of the radius, or,

$$M_T \propto r^4.$$

Finally, collecting the results for the moment of torsion,

$$M_T = c\frac{r^4\theta}{l},$$

where c is a constant not yet determined, but which may be shown to be $n\frac{\pi}{2}$, where n is the coefficient of simple rigidity.

This law may be verified by observing the periods of torsional vibration of wires differing only in their dimensions, from which periods the values of n may also be calculated.

The elasticity of helical springs, so often used in mechanism, is due to a torsion of the wire, at least in all cases where the deformation of each coil is small with respect to its diameter.

When a rod is bent, the layers on one side of the axis in the plane of flexure are in tension, and on the other side in compression. The elasticity is of the sort measured by Young's Modulus.

79. Values of the Coefficients of Elasticity. — The values of the coefficients for certain specimens of common substances are given in the following table.

Since the dimensions of strain are L°, those of the coefficients are the same as for stress, *viz.* $ML^{-1}T^{-2}$. The unit is accordingly $\frac{1 \text{ gm.}}{1 \text{ cm. } 1 \text{ sec.}^2}$.

COEFFICIENTS OF ELASTICITY.

SUBSTANCES.	VOLUME ELASTICITY $= k$.	SIMPLE RIGIDITY $= n$.	YOUNG'S MODULUS $= M$.
Water . . .	2.2×10^{10}	—	—
Mercury . .	2.6×10^{11}	—	—
Glass	$4.1 \times$ "	2.4×10^{11}	6.0×10^{11}
Brass, drawn .	$10.8 \times$ "	$3.7 \times$ "	$10.8 \times$ "
Steel	$18.4 \times$ "	$8.2 \times$ "	$21.4 \times$ "
Wrought Iron .	$14.6 \times$ "	$7.7 \times$ "	$19.6 \times$ "
Cast Iron . .	$9.6 \times$ "	$5.3 \times$ "	$13.5 \times$ "
Copper . . .	$16.8 \times$ "	$4.5 \times$ "	$12.3 \times$ "

80. Viscosity. — Matter in all states shows a gradual yielding to forces which tend to change its form. This property is termed viscosity, and may be defined quantitatively as the

$$\frac{\text{tangential force per unit area}}{\text{shear per unit time}}.$$

Molasses and heavy oils are examples of viscous liquids. On the other hand, alcohol and sulphuric ether are examples of very mobile ones.

81. Impact. — When two bodies collide, their velocities are changed in a manner depending on their momenta and the forces of restitution which accompany the distortion of the bodies. Let m_1 and m_2 be the masses of two smooth spheres moving in the line joining their centers, with velocities v_1, v_2 before impact, and u_1, u_2 after impact. Since whatever change of motion occurs is produced by the same force acting on each during the time they are in contact,

$$m_1(v_1 - u_1) = - m_2(v_2 - u_2), \tag{4}$$

or

$$m_1 v_1 + m_2 v_2 = m_1 u_1 + m_2 u_2; \tag{5}$$

a result which might have been obtained at once from the principle that quantity of motion of a system does not change except by application of forces external to it.

If the bodies are supposed inelastic, or without stresses tending to separate them after collision, the final velocities are the same; hence,

$$u_1 = u_2 = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}. \tag{6}$$

If, however, the bodies are perfectly elastic, they will separate with the same amount of kinetic energy they possessed before impact, and

$$\tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2 = \tfrac{1}{2} m_1 u_1^2 + \tfrac{1}{2} m_2 u_2^2; \tag{7}$$

solving equations 5 and 7,

$$u_1 = \frac{2 m_2 v_2 + v_1(m_1 - m_2)}{m_1 + m_2}, \tag{8}$$

and

$$u_2 = \frac{2 m_1 v_1 - v_2(m_1 - m_2)}{m_1 + m_2}. \tag{9}$$

A simple special case is that in which both balls are of the same size, and one of them is at rest.

Substituting $m_1 = m_2$
and $v_2 = 0$,
in equation 6,

$$u_1 = u_2 = \tfrac{1}{2} v_1, \tag{10}$$

or both balls move after impact with half the velocity of the first before impact.

If the same substitution be made in equations 8 and 9,

$$u_1 = 0,$$

and

$$u_2 = v_1; \tag{11}$$

that is, the balls exchange velocities.

The laws of impact of elastic bodies may be experimentally demonstrated for the simple cases by the apparatus shown in Fig. 71, in which two ivory spheres are suspended so as to swing over a graduated arc. The velocities acquired by a ball in falling are proportional to the square root of the vertical hight, or, in other words, to the sagitta of the arc traversed. But the sagittas are proportional to the squares of the distances of the extremities of the arcs from the vertical through the center of the circle. Hence, within the limits of the apparatus, the velocity of a ball in passing the vertical may be taken as proportional to the arc traversed.

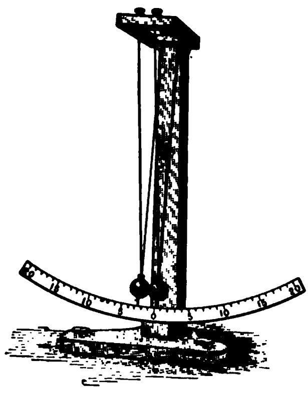


FIG. 71.

For the study of inelastic impact the balls may be rendered practically inelastic by sticking a piece of wax to the face of one of them.

82. Ballistic Pendulum. — The velocity of a projectile may be determined by means of a suspended block of wood, into which the shot is fired.

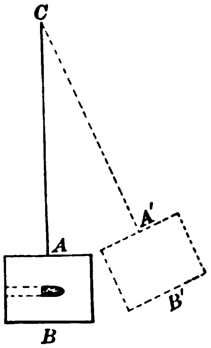


FIG. 72.

Let AB (Fig. 72) represent a block suspended from C, and call its mass M. Let the mass of the shot be m, then as this system fulfills the condition of inelastic impact, equation 6 gives

$$u_2 = \frac{m}{M+m} v_1. \tag{12}$$

An observation of the vertical rise h of the point A will give u_2 by the formula $u_2 = \sqrt{2gh}$, whence v_1 may be found at once from equation 12.

EXAMPLES.

1. A piece of brass wire 0.1066 cm. in diameter and 27.1 cm. long is found to be stretched 0.133 cm. by the addition of a load of .454 kilo. What is Young's Modulus for the wire?

$M = 1.016(10)^{10}$ gms. / cm. sec.^{2}.

2. If a wire 76 cm. long have a period of torsional vibration of 7.52 sec., and a wire 38 cm. long of the same diameter have a period of 5.32 sec., how does the moment of restitution depend on the length of the wire, assuming that the vibrations follow the harmonic law?

Ans. $M_T \propto l^{-1}$.

3. If the period of torsional vibration of a wire 0.092 cm. in diameter have a period of 7.55 sec., and that of a wire of the same length but 0.069 cm. in diameter have a period of 13.15 sec., what is the relation of the moment of restitution to the diameter of the wire?

Ans. $M_T \propto d^4$.

4. A bullet weighing 25 gms. is shot into a suspended block of wood weighing 2.7 kilos. If the block rises a hight of 38 cm., required the velocity of the bullet. $v = 2.97(10)^4$ cm. / sec.

5. A mass of 50 pounds moving at a velocity of 10 ft. per second overtakes a mass of 25 pounds moving with a velocity of 6.0 ft. per second. If both bodies are perfectly elastic, what will be their velocities after impact? *Ans.* 7.3 ft. per second; 11.3 ft. per second.

6. Show that if two equal and elastic spheres collide directly, they will interchange velocities.

7. If a mass of 625 gms. moving with a velocity of 786 cm. per sec. meet a mass of 164 gms. moving in the opposite direction with same speed, what will be their velocity after impact, supposing the bodies to be inelastic? $u = 459$ cm. / sec.

CHAPTER VI.

MECHANICS OF FLUIDS.

83. Pressure in a Fluid at Rest. — Let Fig. 73 represent a vessel containing a liquid at rest, and suppose a small cylinder with vertical sides to become rigid.

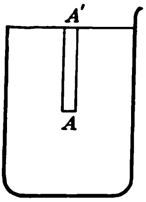


FIG. 73.

It will be in equilibrium under the action of the pressures on the sides, the pressure on the bottom, and its weight.

Now, by the definition of a fluid, the pressures on the sides must be normal; hence, they form a balanced system, and do not affect the weight of the cylinder. Therefore, the weight and upward force on the bottom must be in equilibrium.

Let $h =$ hight of the cylinder,
$A =$ the area of the base,
$\rho =$ the density of the liquid,
and $p =$ the pressure at the point considered;

then

$$hA\rho g = pA,$$

or,

$$p = h\rho g, \tag{1}$$

which must be equal in all directions, by the definition of a fluid.

If the upper surface A' were itself subject to a pressure, P, as of the air, the pressure at A would be increased by that amount.

84. Surfaces of Constant Pressure. — Surfaces of constant pressure in a fluid are at right angles to the acting forces. Let Fig. 74 represent a portion of a fluid in equilibrium, and ab a portion of a surface of equal pressure. Suppose cd, a small cylinder of the fluid, having its axis on the line of equal pressure, to become rigid. The forces acting on the cylinder are the pressures on the ends and the pressures on the sides, to which may be added some external force, F.

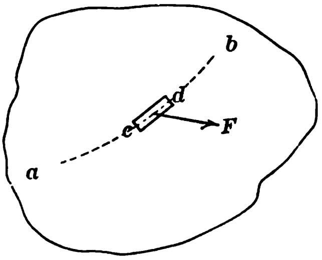


FIG. 74.

Since cd is a line of equal pressure, the forces due to the pressures on the ends are equal and opposite.

The pressures on the sides are perpendicular to the surface, since they are exerted by a fluid.

The resultant force is therefore perpendicular to the axis, and in consequence the force F, which is equal and opposite to this resultant, must also be at right angles to cd. But cd is any line in the surface of equal pressure, therefore F is perpendicular to this surface.

In the case of a liquid acted on by weight only, any small free surface must accordingly be a horizontal plane, and by means of the preceding article it may be shown that any horizontal plane in the liquid is a surface of equal pressure.

When several fluids which do not mix are superposed, they will arrange themselves in order of increasing density from top to bottom, to fulfill the condition of minimum potential energy, and the surfaces of separation will be horizontal, since they are surfaces of equal pressure.

85. Levels. — A spirit-level consists of a tube (Fig. 75) bent to the arc of a circle, nearly filled with alcohol, and

hermetically sealed. The small air bubble will always stand at the highest point on account of its lesser density. When mounted in a suitable base, the level may be used to set a line horizontal or vertical.

Call the upper inner radius of the tube R; then if, when the inclination of the level is changed an angle, ϕ, the bubble moves a distance, d,

$$\phi = \frac{d}{R}.$$

If, for instance, R were 60 inches, a variation of 6 min. in the inclination of the level would produce a deflection of $\frac{1}{10}$th of an inch in the bubble. A spirit-level 3 inches long would in this case indicate a deviation as definitely as a plumb line 5 feet long. Such levels may be made of any degree of sensitiveness, though the precision of their reading is considerably limited by the accuracy with which it is possible to form the inner surface of the tube to a true arc of a circle.

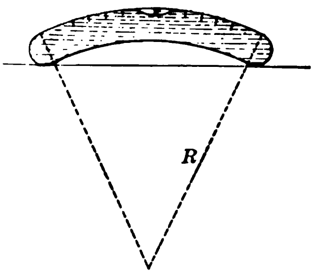


Fig. 75.

When the upper surface of the vessel is made spherical, the instrument is known as a box-level, and may be used for leveling a horizontal surface. As usually made, the box-level is not serviceable, on account of the readiness with which the alcohol evaporates.

86. Principle of Archimedes. — The principle of Archimedes may be stated as follows: a body immersed in a fluid at rest loses a portion of its weight equal to the weight of its own volume of the fluid.

To prove it, suppose any portion, A, of a fluid at rest (Fig. 76) to be separated from the remainder by a closed surface wholly immersed. Since this portion by supposition is at rest, the pressure of the fluid on the outside of the surface must be such as to sustain the weight of the fluid within.

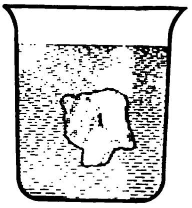


FIG. 76.

Now, if the fluid within this imaginary surface be replaced by any other substance, it is manifest that the pressure of the fluid outside will not be modified. But this pressure caused the volume of the fluid enclosed to lose its weight. Therefore, any other body of the same shape would lose the same amount, that is, the weight of its own volume of the fluid.

Call m the mass of the body,
m_1 the apparent mass as determined by weighing in the fluid,
V the volume of the body,
ρ' the density of the fluid;

then Archimedes' Principle may be written,

$$mg - m_1 g = V\rho' g. \tag{2}$$

In all but the most refined observations the loss of weight, on account of immersion in air, may be neglected. If the body be immersed in a fluid whose density is greater than its own, the loss of weight is greater than the weight of the body. In such a case the body is acted on by an upward force, and if not constrained will rise till the weight of the volume of the fluid displaced is equal to the weight of the body.

87. Determination of Density. — When the volume of a solid is known, its density may be calculated directly from the definition

$$\rho = \frac{m}{V}.$$

If the body is of such shape that its volume may not be readily calculated, the density may be found by weighing the body in a liquid of known density.

Thus, writing $V = \frac{m}{\rho}$ in equation 2, and solving for ρ,

$$\rho = \frac{m}{m - m_1}\rho'. \tag{3}$$

If the liquid used is water, its density may be taken as unity, and the numerical value of the density of the body will be found by dividing the weight in air by the loss of weight in water.

When the body is less dense than the fluid, it is necessary to use a sinker. Let m_2 be the apparent mass of the sinker when weighed in the fluid, and M the apparent mass of both sinker and body in the fluid; then, as the loss of weight is proportional to $m + m_2 - M$, the density is given by

$$\rho = \frac{m}{m + m_2 - M}\rho'. \tag{4}$$

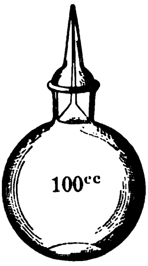


FIG. 77.

The density of a liquid may be obtained by the use of a flask (Fig. 77) so constructed that when its stopper is in place its content is exactly 100 cc. This condition is most readily effected by first making the glass stopper too long or the volume of the flask too small. Then, by grinding off the end of the stopper, the required volume may be secured as closely as may be

desired. In order to render the complete filling easy, the stopper is provided with a small hole through which the excess of liquid in the flask escapes when the stopper is pressed in. The difference between the weight of the flask when full and when empty gives the mass of the liquid. The density is obtained by dividing this mass by the volume of the flask. The density of a gas may be found in an entirely analogous manner. However, on account of the relatively small density of gases, certain precautions are necessary. For example, the flask should be as large as is convenient, in order that its mass may not be too great compared to that of its contents; and again, when the flask is weighed empty it should be exhausted, as nearly as is possible, of every kind of matter.

FIG. 78.

88. Areometer. — The areometer is an instrument designed to indicate the density of a liquid by the depth at which it will float. It consists essentially of a weighted glass bulb with a graduated stem (Fig. 78). Instruments intended for use in liquids lighter than water read upward from the bulb, while those designed for denser liquids read from the top down. The scale is not infrequently arranged so as to give the percentage of one of the components in a solution. Thus, for instance, the areometer is often used to determine the concentration of commercial alcohol, or sulphuric acid, and the comparative richness of various samples of new milk. In the arbitrary scale of Baumé the point to which the instrument sinks in water is marked 0°, and the immersion in a solution of 15 parts common salt to 85 parts water, 15°. For liquids lighter than water the point of immersion in a solution of 10 parts water to 90 parts salt is marked 0°, and the immersion in pure water 10°.

TABLE OF DENSITIES.

UNIT 1 gm. per cc.

Solids.

Aluminium	2.6	Ice	0.917
Antimony	6.7	Iron (cast)	7.0 to 7.6
Bismuth	9.8	" (wrought)	7.3 to 7.8
Brass	8.4	Lead	11.3
Brick	2.1	Nickel	8.9
Clay	1.9	Oak	0.7 to 1.0
Copper	8.9	Platinum	21.5
Cork	0.24	Quartz	2.65
Diamond	3.5	Sand (dry)	1.4
Gas Carbon	1.9	Silver	10.5
Glass (crown)	2.5 to 2.7	Sodium	0.98
" (flint)	3.0 to 6.3	Sulphur (native)	2.0
Gold	19.3	Tin	7.3
Graphite	2.3	Zinc	7.1

Liquids at 0° C.

Alcohol	0.806	Nitric Acid	1.56
Bisulphide of Carbon	1.29	Oil (linseed)	0.94
Chloroform	1.53	" (mineral)	0.76 to 0.83
Ether	0.736	" (olive)	0.91
Glycerine	1.27	Sea Water	1.03
Hydrochloric Acid	1.27	Turpentine	0.87
Mercury	13.6		

Gases at 0° C., and 760 mm. at latitude 45°.

Air	0.001293	Marsh Gas	0.000715
Carbon Dioxide	0.001965	Nitrogen	0.001254
Chlorine	0.003167	Oxygen	0.001429
Hydrogen	0.0000895		

Gases at 0° C., and a pressure of 10^6 dynes per square centimeter.

Air	0.001276	Marsh Gas	0.0007173
Carbon Dioxide	0.001951	Nitrogen	0.001239
Chlorine	0.003091	Oxygen	0.001411
Hydrogen	0.00008837		

Density of Saturated Water Vapor.

0°	0.00000475 gm. / cc.
10°	0.00000922
20°	0.0000170
30°	0.0000301
40°	0.0000509
50°	0.0000829
60°	0.0001306
70°	0.0001994
80°	0.0002959
90°	0.0004284
100°	0.0006062
Air at 100°	0.0009459.

89. Barometer. — The pressure of a gas is conveniently determined by comparison with the pressure exerted by a known column of liquid. Thus, if a tube of glass, closed at one end and more than 76 cm. long, be filled with mercury and inverted in a vessel of the same liquid, the pressure of the air will support a column of the mercury about 76 cm. high. This pressure, $76 \times 13.6 \times 980 = 1.013(10)^6$ dynes per square centimeter, is sometimes taken as a unit, and is called an *atmosphere*.

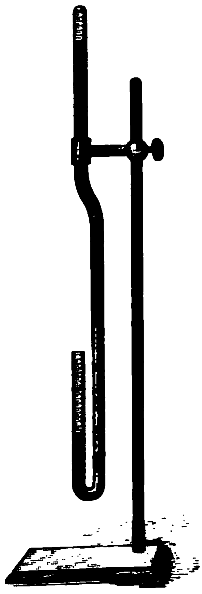


FIG. 79.

Instruments used to measure the pressure of the air are known as barometers. A common form, called the siphon barometer, consists of a glass tube closed at the upper end, and bent into a U-form at the lower end, as shown in Fig. 79. The distance between the bend and the closed end must be considerably more than 76 cm. Special precautions also are necessary to assure that the

mercury and the tube are perfectly clean and that no air is included. The difference of level between the free surfaces of the mercury may be read from the millimeter graduations on both legs of the tube.

90. Fortin's Barometer. — A portable form of barometer devised by Fortin is shown in Fig. 80. The cistern *C* is closed at the bottom by a leather bag, *B*, which may be raised and lowered by a screw, *S*. A glass cylinder, *G*, near the top permits the upper surface of the mercury to be seen and brought in contact with the point *P*, which marks the zero of the barometer scale. The tube is attached to the cistern by means of a piece of chamois skin, *F*, which prevents the escape of the mercury but does not support any difference of pressure between the outside and inside air. The glass tube is protected by an outer brass case graduated to millimeters and furnished with a vernier at the top, the whole being arranged so that the barometer may be mounted in gimbals on a tripod.

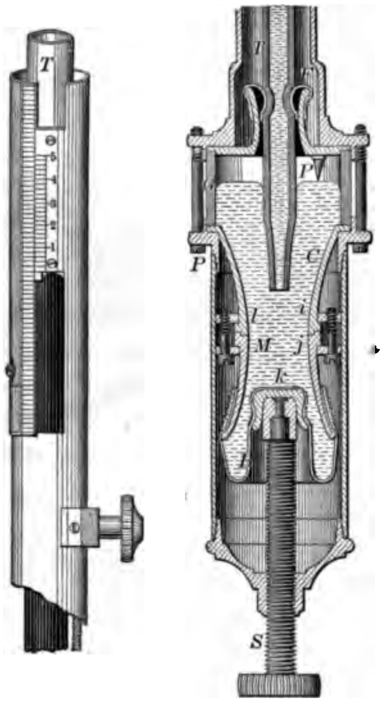


FIG. 80.

When the barometer is to be transported, the screw *S* is raised till the cistern and tube *T* are entirely full of mer-

cury. Entrance of air into the tube and breaking of the glass by bumping of the mercury against the end are thus prevented.

To obtain the true hight of the barometer the observed reading must be corrected for temperature and capillarity.

91. Aneroid Barometer. — The pressure of the air may also be indicated by the deformation it produces in a metal spring. An instrument designed for this purpose is called

FIG. 81.

an *aneroid* barometer. In the usual type (Fig. 81) it consists of a metal box, *B*, exhausted of air, with a top corrugated in circular ridges to give it greater flexibility, but prevented from collapsing by a stiff spring, *R*, attached at *M*. The motion of the cover as it rises and falls, with varying pressure, is transmitted by means of a system of levers, *l*, *m*, *t*, and a chain, *c*, to a horizontal pointer, *P*. When in use an empirically graduated scale is placed below the pointer so that the reading gives the hight of the mercury column for the corresponding pressure. The advantages of such an instrument are its portability and sensitiveness. It possesses

one serious drawback which impairs its usefulness as an instrument of precision: the spring is liable to anomalous variations which can only be detected by comparison with a mercurial barometer. Such variations are most likely to occur when the instrument has been subjected to widely different pressures. Temperature changes are usually compensated by a special device in the instrument itself.

92. Measurement of Altitude by the Barometer. — In the case of a liquid which may be treated as incompressible, that is, having everywhere the same density, the difference of pressure between two points was shown (Art. 83) to vary directly as the vertical distance between these points.

When, as in the case of the atmosphere, the density of the fluid is variable, the difference of elevation of two stations may be calculated from the observed values of the air pressure.

Let P = air pressure at the lower station,
p = " " " " upper station,
θ = temperature Fahrenheit,
h = vertical distance in feet between stations;

then it may be shown that

$$h = \left\{ 60360 + (\theta - 32°)\,122.68 \right\} \log_{10} \frac{P}{p}.$$

93. Barometer as a Weather Glass. — The pressure of the air at any place is so influenced by meteorological conditions that it is possible, from observations of the barometer, to predict with more or less certainty coming changes in the weather. Thus, a rapid fall of the barometer is usually followed by a storm, and a rising barometer by fair weather. The words Rain, Set Fair, etc., frequently marked opposite

certain readings on a barometer, are without significance in connection with those pressures alone.

94. Manometer. — When it is desired to determine the pressure of a gas contained in a closed vessel, use is often made of a bent tube partly filled with mercury. If the pressure to be determined does not much exceed that of the air, the end C of the manometer tube, as it is called, is left open, as in Fig. 82A. The end D being put in communication with the vessel, the required pressure will then be the reading of the barometer increased by the difference in level between L and M. If, however, the pressure of the gas in the vessel is considerable, the end of the tube is closed, as in Fig. 82B, and its amount is read from an empirically graduated scale placed at the side of the tube.

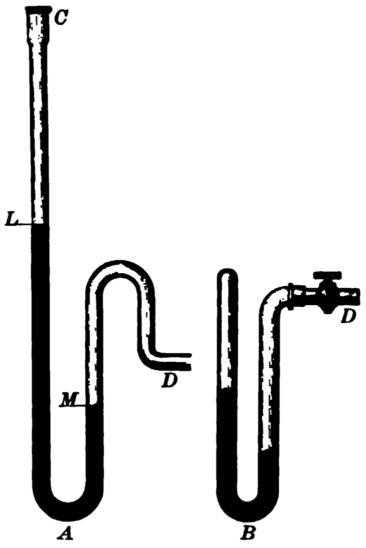


FIG. 82.

95. Bourdon Pressure-Gauge. — In the mechanic arts a metallic gauge devised by Bourdon is extensively used for the measurement of pressures.

It consists of a thin metal tube, of elliptic or flattened cross section, bent into the form of a circle (Fig. 83).

An increase of the pressure within such a tube expands it with reference to the cross section, but causes a diminution

of the curvature in a plane at right angles. The motion of the tube in straightening is communicated by means of a toothed segment, *R*, and pinion, *K*, to a pointer, *P*. The pressure may be read in any desired unit from a dial beneath the pointer, the graduations of which have been empirically determined.

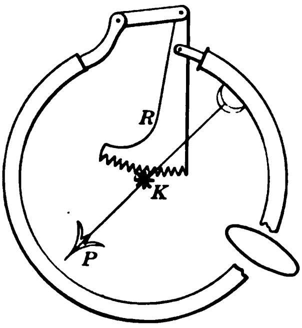


FIG. 83.

An instrument of this sort, arranged to measure pressures below one atmosphere, is known as a Vacuum Gauge. It is commonly graduated to read "inches of mercury."

The straightening of a bent pipe, utilized by Bourdon in the manometers just described, is exhibited on a larger scale by any coiled rubber tube when filled with water under pressure.

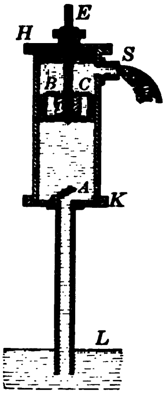


FIG. 84.

96. Pumps. — A pump is an instrument intended to transfer a fluid under pressure from one vessel to another.

A form designed to elevate liquids subject only to the pressure of the atmosphere is shown in Fig. 84. *D* is a close-fitting piston which moves up and down in the pump barrel *HK*. *A*, *B*, and *C* are valves opening upward.

Suppose, for instance, that the piston is moving upward. Then the weight of the fluid on *B* and *C* keeps these valves closed while the mass is being lifted and discharged through the spout *S*. At the same time the valve *A*, which is acted on from below by the pressure of

the air diminished by the pressure due to the weight of the column of the liquid LK, will be forced open, allowing the portion of the barrel below D to fill. On the down stroke of the piston, B opens, equalizing the pressure above and below, while A is closed by the weight of the liquid.

Since the density of mercury is 13.6 times that of water, the column of water which the air under ordinary conditions will sustain is $30 \cdot \text{in.} \times 13.6 = 34$ feet, but a pump of the

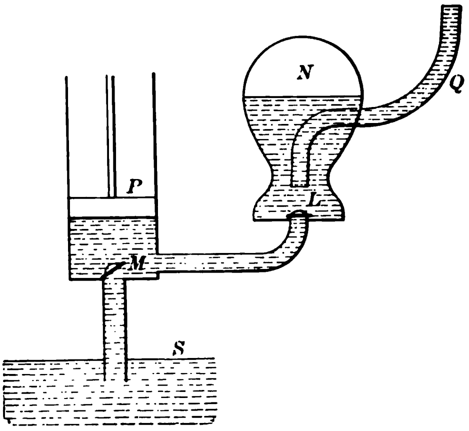


FIG. 85.

type just described will not work effectively at a hight of more than 25 feet. However, it is possible by a different arrangement of the valves to force the liquid to a hight limited only by the ability of the materials to withstand the pressures imposed.

The operation of the force pump may be understood from inspection of Fig. 85. P is a solid piston or plunger. M is a valve which is raised, during the upward stroke, by the pressure of the air on the surface S a short distance below it. L is a valve opening during the downward stroke to admit

the liquid to the chamber N, whence it is forced out through the discharge pipe Q by the pressure of the air confined in the upper part of the chamber.

97. Air Pump. — The pumps used to exhaust a vessel of air do not differ materially in their mode of operation from the one shown in Fig. 84 for the transfer of liquids.

Let R (Fig. 86) be the vessel from which it is desired to exhaust the air. HK is the pump barrel, D the piston, and S a valve opening upward when the pressure below D is

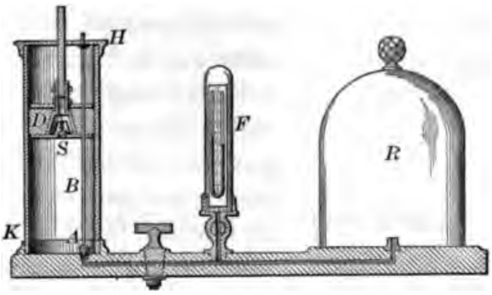


FIG. 86.

greater than that above. In this case it is necessary, however, that the valve A shall have a positive motion with each stroke of the piston. This is effected by fastening to the conical plug A a rod, B, which slides through D with a certain amount of friction, so that A moves a short distance up or down with it at the beginning of the stroke before striking certain stops provided to limit its travel.

When the piston rises, the air in the receiver R expands and fills the barrel HK; at the beginning of the down stroke A closes and the air in the pump is forced out through B during the descent of the piston. Thus, at each stroke a certain fraction of the air remaining in R is removed, but

on account of leakage and other imperfections, the pressure cannot be reduced much below one millimeter of mercury.

Double-acting pumps of this type are much used where the quantity of gas to be removed is considerable. If a high degree of exhaustion is desired, use is made of one of the mercury pumps described in Arts. 102, 103.

98. Siphon. — A siphon is the name given to a bent tube used to transfer a liquid from one vessel to another.

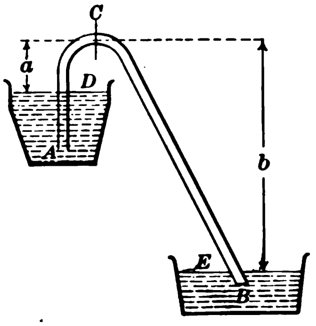


FIG. 87.

Suppose a tube, ACB, after being filled with the liquid, is inverted so that each end is immersed as in Fig. 87.

Call a and b the respective hights above D and E of any point in a vertical section through the highest part of the tube, and let h be the hight of the column which would be supported by the air. Then the pressure toward the right, at any point in this section, is proportional to $h - a$. Likewise toward the left it is proportional to $h - b$. Each particle of the fluid in this section is accordingly urged to the right with a force which is proportional to

$$(h - a) - (h - b) = b - a = \text{constant}.$$

The liquid will accordingly flow from A to B provided a is not greater than h.

99. Velocity of Efflux. — Let C (Fig. 88) be a vessel filled with a liquid. Call the hight AB, h, and suppose a small quantity, m, of the liquid to be drawn off at the level A, slightly lowering the surface B. If the quantity m be raised

the hight h and returned at the top, the work mgh will be done, and the former level of B will be restored. It is further evident that if the quantity m were replaced in any other way, as *e.g.* by forcing it in through the orifice A against the pressure at that level, the same amount of work must be done, for the final value of the potential energy of the whole mass with respect to the earth is the same in each case.

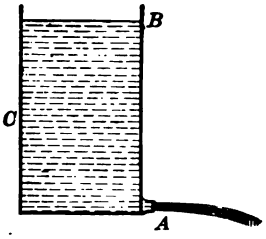


FIG. 88.

Therefore, neglecting viscosity, and calling the velocity of efflux v, the kinetic energy $\frac{1}{2}mv^2$ of the escaping mass must equal the diminution of the potential energy of the liquid in the vessel, or,

$$v^2 = 2gh. \tag{5}$$

This equation, known as Torricelli's theorem, was announced long before the recognition of the law of conservation of energy which makes its demonstration so simple.

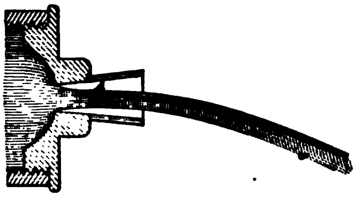


FIG. 89.

This result must not, however, be used to calculate the amount of discharge from the area of the orifice, because the size of the issuing jet is considerably modified by the shape of the opening. As shown in Fig. 89, the converging of the lines of flow produces a contraction in the area of the issuing stream at A, the value of which can, in general, be determined only by experiment.

100. Reduction of Pressure at the Side of a Moving Stream. — When a column of a fluid is in motion, the pressure at any point in its side is less than if the fluid were at rest. If ρ

is the density of the fluid, and p the amount of this diminution, the value of the latter may be calculated by the aid of equation 5 thus:

$$p = hg\rho = \tfrac{1}{2}v^2\rho. \tag{6}$$

It is necessary to observe, however, that equation 6 is only a first approximation, because the influence of viscosity has been neglected.

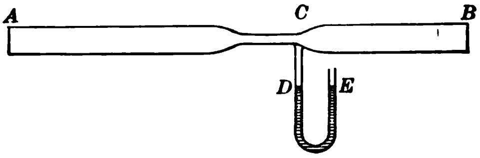


FIG. 90.

This reduction of pressure may be readily demonstrated by the apparatus shown in Fig. 90. AB is a glass tube with a constriction at C bearing a short U-tube in communication with the interior, and filled with water to the level DE. When air is blown through AB, the level at E falls and that of D rises, showing a reduction of pressure at the point C.

The same thing may be shown in another way by the apparatus illustrated in Fig. 91, which consists of a tube, G, terminated by a disc, A. Suspended from this by wires, on which it may slide up and down, is a light vulcanite disc, F. When air is smartly blown through the pipe G, the pressure in the space between the discs is so reduced that F is raised by the air pressure on the under side.

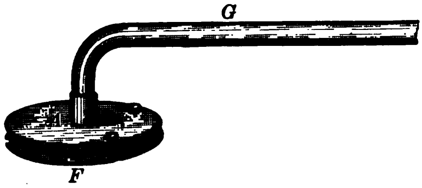


FIG. 91.

101. Jet Pump. — The principle just presented has been utilized in the construction of various pumps, of which Fig. 92 may serve as the representation of a widely used type.

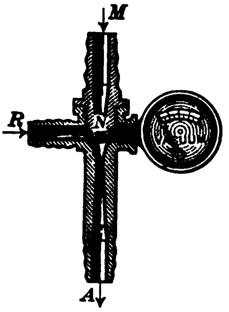


FIG. 92.

If a stream of fluid be forced through the pipe *M*, a sufficient reduction of pressure may be secured at the side of the escaping jet *N* to permit the inflow at *R* of the same or any other fluid.

The forced draft on a locomotive is obtained by what is virtually a pump of this type.

102. Sprengel Air Pump. — A very efficient air pump, devised by Sprengel, is shown in Fig. 93. *C* is a reservoir of mercury supported on a movable shelf, *A*, and connected through *I* and *S* to a vertical glass tube, *F*, about 2.5 millimeters in diameter. Connection between *F* and the vessel to be exhausted at *R* is made through a bulb, *E*, containing sulphuric acid designed to absorb any water vapor that may be present. *B* is a manometer to indicate the degree of exhaustion attained.

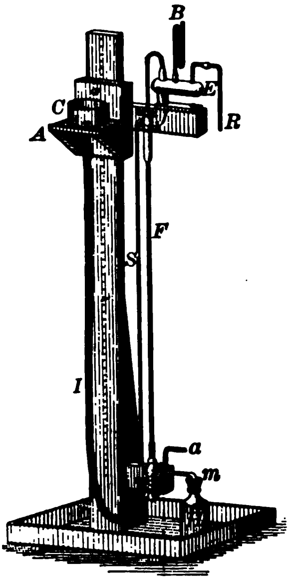


FIG. 93.

When the mercury is allowed to flow through the tube *F* in a broken stream, there is a reduction of pressure at *J*, due in part to the velocity of the moving fluid, and in part to the fact that each drop of the mercury acting like a piston

is the density of the fluid, and p the amount of this diminution, the value of the latter may be calculated by the aid of equation 5 thus:

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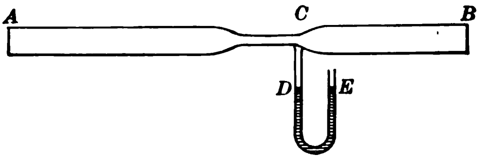

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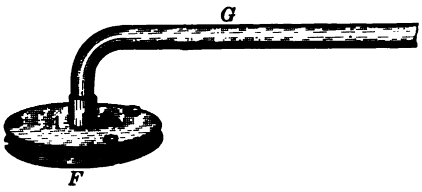

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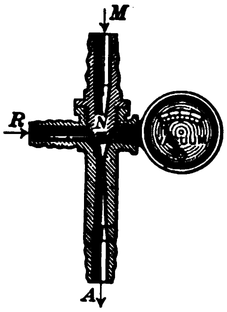

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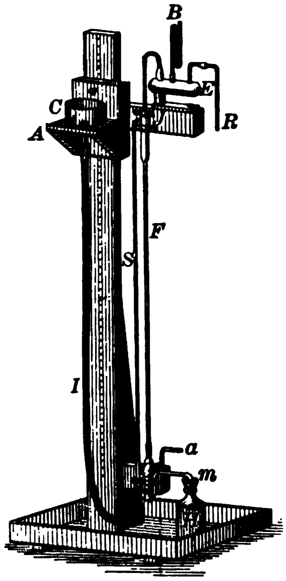


FIG. 93.

When the mercury is allowed to flow through the tube *F* in a broken stream, there is a reduction of pressure at *J*, due in part to the velocity of the moving fluid, and in part to the fact that each drop of the mercury acting like a piston

105. Support of a Ball on a Jet of Fluid. — Let B (Fig. 97) be a ball supported at the side of a jet of fluid. It may be regarded as acted on by three forces, namely: W, the weight of the ball; P, an excess of air pressure on one side due to the velocity of the moving stream; and I, the force of impact of the fluid against the ball. If these forces form a closed triangle, the ball will be in equilibrium. A position of the ball may, in general, be found for which the equilibrium will be stable.

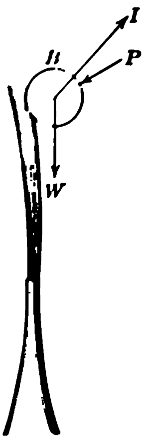


FIG. 97.

EXAMPLES.

1. What is the pressure at a depth of 76.3 cm. in a pool of mercury?
Ans. $1.017(10)^6$ dynes / cm.^{2}.

2. A U-tube is partly filled with water. How many inches of oil having a density of 0.79 gm. per cc. must be added in order to raise the water 4.5 in. in one leg above its first level?
Ans. 11.4 inches of oil.

3. What is the density of a body whose mass is 678 gms., if it weighs 235 gms. when immersed in a fluid whose density is 1.94 gms. per cc.? $\rho = 2.97$ gms. / cc.

4. A wire 12.6 cm. long and containing 435 gms. weighs 299 gms. when immersed in water. What is the mean diameter of the wire?
$d = 3.71$ cm.

5. A body having a density of 2.35 gms. per cc. weighs 624 gms. when immersed in a liquid whose density is 0.827 gm. per cc. What is the mass of the body? $m = 963$ gms.

6. A piece of wood containing 46.7 gms. is immersed in water by the aid of a sinker which weighs in water 75.8 gms. The combined weight of the wood and the sinker is 32.9 gms. What is the density of the wood? $\rho = 0.521$ gm. / cc.

7. If the density of ice is 0.918 gm. per cc., and that of sea water is 1.03 gms. per cc., what is the volume of an iceberg exposing 697 cu. yds.? *Ans.* 6400 cu. yds.

8. An iron body weighing 275 gms. floats in mercury with 0.556 of its volume immersed. Required the volume and density of the body.
$\rho = 7.56$ gms. / cc.; $v = 36.4$ cc.

9. If the apparent mass of a body when weighed in a fluid of density ρ' is m', and when weighed in a fluid ρ'' is m'', what is the mass of the body? $= \dfrac{m''\rho' - m'\rho''}{\rho' - \rho''}$.

10. A mass of 28.1 gms. having a density of 5.59 gms. per cc. and a mass of 35.8 gms. weigh the same when immersed in water. What is the density of the second body? $\rho = 2.81$ gms. / cc.

11. A sphere having a density 0.957 gm. per cc. and a volume of 168 cc. floats in a vessel of water. If a layer of oil having a density of 0.892 gm. per cc. is poured on the water so as to cover the sphere, how much of the latter will be immersed in the water? *Ans.* 101 cc.

12. A cylinder of cork h cm. high, having a density ρ, floats on a liquid of density ρ'. If the air above the liquid be removed, show that the cylinder will sink a distance $\dfrac{\rho''(\rho' - \rho)}{\rho'(\rho' - \rho'')} \cdot h$ where ρ'' is the density of the air.

13. An ornament made of an alloy of gold and silver weighs 76.8 gms., and has a density of 18.0 gms. per cc. Assuming that the volume of the alloy is equal to the combined volumes of the components, find the amount of gold and of silver in the body.
Ans. 70.2 gms. gold; 6.6 gms. silver.

14. If a mass, m_1, having a density, ρ_1, weighs the same as a mass, m_2, when both are immersed in a fluid of density ρ', prove that the density of the second body is $\rho_2 = \dfrac{m_2\rho'}{\dfrac{m_1\rho'}{\rho_1} + m_2 - m_1}$.

CHAPTER VII.

SURFACE TENSION.

106. Phenomena at the Surface of a Liquid. — The forces which are considered as effective in holding together the particles of a liquid or solid are termed molecular forces, since they are sensible only at insensible distances. When two fractured pieces of porcelain are placed together, even

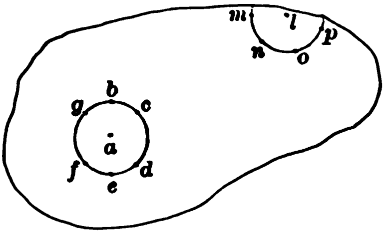


FIG. 98.

though the contact be so close that the break is imperceptible to the eye, there is no tendency for them to unite, because the parts are still separated by a distance greater than the molecular range. Two surfaces of glass, accurately plane and very clean, when pressed together will sometimes give evidence of a partial union. If, however, two freshly scraped surfaces of a more yielding substance, such as lead, be pressed together, a marked tendency to weld may be observed. In the case of liquids, these molecular forces give rise to a phenomenon at the surface which resembles in many respects the tension in a stretched membrane. Thus, suppose *a* (Fig. 98) represents a molecule within a body of fluid, and *b*, *c*, *d*, *e*, *f*, *g* a series of molecules at equal distances from *a*, but within the range of molecular forces. It is obvious that *a* would, on the whole, be attracted no more in one direction than another, and hence would be in equilibrium under these forces. If *l*, on the other hand, be a particle which is nearer to a free surface than the distance of

molecular range, then it is evident that the resultant of the attractions of the neighboring particles m, n, o, p will urge it toward the interior of the body. Accordingly, since liquids have no shearing elasticity, the shape assumed by the body will be that for which the area is the least consistent with the given volume and the boundary conditions.

The phenomena of surface tension in extended masses of liquid are usually greatly masked by the distorting effect of weight. This influence of weight on the surface of a liquid body may be eliminated by immersing the body in another liquid of its own density. Experiments in this mode are conveniently performed upon olive oil immersed in a mixture of alcohol and water. The density of this oil is about $0.917 \frac{\text{gm.}}{\text{cc.}}$. Hence, as alcohol, whose density is $0.806 \frac{\text{gm.}}{\text{cc.}}$, unites with water in all proportions, it is easy to prepare a solution in which the oil will float at any depth. If a small quantity of oil be introduced into such a solution by means of a pipette, it will assume a spherical form in accordance with the familiar fact that the sphere has a smaller area than any other surface enclosing an equal volume. Another method of avoiding the disturbance of weight is to use very thin films, such as are readily produced from a solution of soap and water. As the mass of liquid involved is quite small, its weight will exercise but little influence on the form of the film.

Though the surface of a liquid behaves, in general, like a stretched elastic membrane, it nevertheless differs from such a membrane in two important particulars, namely, that liquid films, when unrestrained, contract indefinitely; and, secondly, the tensile force is the same in all directions and independent of the thickness, at least when the latter exceeds a certain very small value.

The former of these statements may be verified by the observation of a soap bubble which has been blown on a funnel, as in Fig. 99. If the small end of the funnel be left open, the sphere will slowly contract, expelling the air until the film has become a plane across the mouth of the funnel. The process does not stop here, however, but the contraction still goes on, this result being accomplished by the rise of the film in the cone until it finally reaches the smallest cross section, provided the film is sufficiently stable.

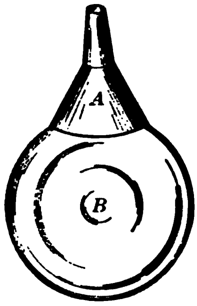


FIG. 99.

The second peculiarity may be exhibited by the following experiment. Let a thread be attached to one side of a wire loop (Fig. 100), and held while the wire is dipped into a soap solution so as to form a film upon it. If the lower portion of the film be now broken, the upper portion of the film, where bounded by the thread, will take an accurately circular form, AB, however it may be changed in position by pulling on the free end of the thread BC. Since the curvature of this arc is constant, it follows that the force is the same in all directions in the surface, and also, since the thickness is not constant, as is shown by the varying colors of the film, it follows that the surface tension is independent of the thickness.

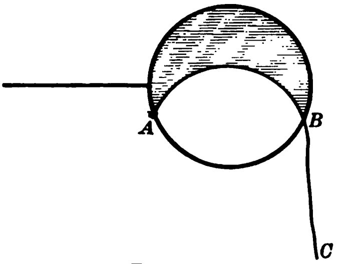


FIG. 100.

107. Change in Pressure in Passing through a Liquid Surface. — Let $ABCD$ (Fig. 101) represent a portion of a plane liquid surface, and suppose that the force F must be applied

across the line ab in order to keep the film stretched. The measure of the surface tension is then defined as the force per unit width of the film, *i.e.*

$$T=\frac{F}{ab}.$$

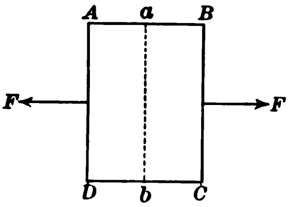


FIG. 101.

Suppose the film now to be bent so as to form a portion of a cylindrical surface of radius r, as in Fig. 102, and that the forces FF are resolved into components parallel and perpendicular to KO. Since each of the former components will be $F \sin\theta$, as is seen from the figure, the total normal force at the center K of the film will be

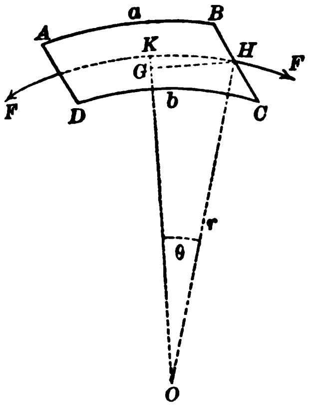


FIG. 102.

$$(1)\quad N=2F\sin\theta=2F\frac{\overline{GH}}{\overline{KO}}.$$

After dividing both sides of this equation by AD, the breadth of the film, equation 1 may be written

$$(2)\quad \frac{N}{2\overline{GH}\cdot\overline{AD}}=\frac{1}{\overline{KO}}\cdot\frac{F}{\overline{AD}}=\frac{T}{r}.$$

If, now, the dimensions of the film be made smaller and smaller, it is evident that $2\overline{GH}\cdot\overline{AD}$ approaches the area of the film, and the left-hand member becomes the pressure just below K, due to the tension of the surface. Calling this pressure p, equation 2 becomes at the limit

$$(3)\qquad p=\frac{T}{r};$$

that is to say, in passing from the convex to the concave

side of a liquid cylindrical surface, the pressure must increase by $\frac{T}{r}$ dynes per square centimeter, on account of the tension in the film.

Equation 3 may be extended to a surface of double curvature thus: Since in the case of a cylindrical surface the pressure due to the film depends on the tension, T, and the curvature, $\frac{1}{r}$, of the line AB (Fig. 102), and nothing else, it follows that if the surface be bent to a curvature, $\frac{1}{r'}$, in the line ab, which is at right angles to CD, the pressure which was due to T and $\frac{1}{r}$ will not be altered except by an increase due entirely to the tension at right angles to AB, and the curvature in the direction ab, *i.e.* by an amount, $\frac{T}{r'}$.

Hence, in passing through a liquid surface of double curvature, the pressure changes by an amount,

$$p = T\left(\frac{1}{r} + \frac{1}{r'}\right), \tag{4}$$

where r and r' are the radii of curvature of two normal sections at right angles to each other.

It should be noted that in applying this equation to a soap film, since there are two liquid surfaces very near together, the value of T is to be taken as twice that due to a single surface.

108. Angle of Contact. — If the molecular attractions of the particles of a solid for those of a liquid are greater than the attractions of the liquid molecules for each other, the liquid when brought into contact with the solid will adhere

to it, in which case it is said to wet the solid. If, on the other hand, the mutual attractions of the molecules for each other exceed those exerted on them by the solid, the liquid does not wet the solid. In general, the surface of the liquid will meet that of the solid at a definite *angle of contact* depending on the nature of the substances actually concerned in the phenomenon.

In the first-mentioned case this angle will be acute if it have any finite value, and in the second case, obtuse.

The angle of contact may be measured in a variety of ways. In the method of Guy-Lussac the liquid is gradually introduced into a sphere from below until it reaches a level where the liquid surface is entirely plane at the edges. The required angle may then be calculated from the depth of the liquid and the dimensions of the sphere.

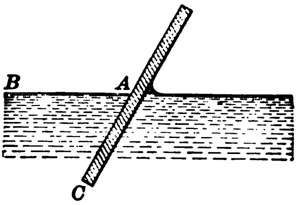


FIG. 103.

Another way is to dip a plate into the liquid, inclining it until the surface of the latter is horizontal at the point of contact, as in Fig. 103. *BAC* will then be the angle required.

If a small amount of liquid be placed on a carefully cleaned surface (Fig. 104), the form of the drop will depend only on the surface tension, the acceleration of weight, the density of the liquid, and the angle of contact.

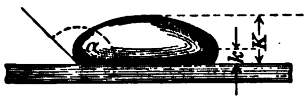


FIG. 104.

This angle, α, may accordingly be calculated from certain observed dimensions of the drop, such as the thickness, K, and the vertical distance, k, between the vertex of the drop and the point on the meridian curve at which the tangent is vertical. In case the liquid wets the solid, a bubble of air may be used, as in Fig. 105.

The angle of contact, as well as the surface tension, varies greatly with the cleanness of the surfaces supposed to be in contact. With water against clean glass the angle is very small, but if the surfaces are in the least contaminated, the angle may reach, or even exceed, 90°. The contact angle of mercury against glass varies from 129° to 143°.

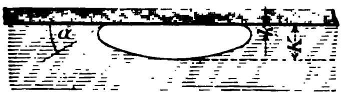


FIG. 105.

109. Capillary Phenomena. — The first phenomenon of surface tension to be observed and studied was the rise of liquids in capillary tubes, so named from their fine, hair-like bore. The term *capillarity* is sometimes used in an extended sense to include all the phenomena of surface tension.

When a tube is inserted in a liquid which wets it, the surface is observed to rise on the outside slightly, and on the inside to a considerable hight above the hydrostatic level, as is shown in Fig. 106, provided the bore is small. The explanation is as follows: Since the liquid wets the tube, the angle of contact is acute, and the curved surface, or *meniscus* as it is called, must be convex downwards with an excess of pressure on the concave side equal to $T\left(\frac{1}{r}+\frac{1}{r'}\right)$. Where the surface is horizontal, as at C and D, the curvature is the same on each side, and the surface tension has no effect on the level. Where the surface, however, has a negative curvature, as at A, the

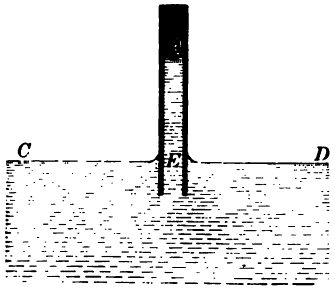


FIG. 106.

liquid rises to such a level that the pressure just below A is that at CD diminished by the pressure due to the hight of the column of fluid AE. When the liquid does not wet the tube, *i.e.* when the curvature of the surface is positive, the opposite effect takes place, the surface B (Fig. 107) sinking to such a depth below the hydrostatic level that the pressure due to the liquid column is equal to that required by the curvature of the surface and its surface tension.

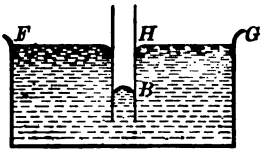


FIG. 107.

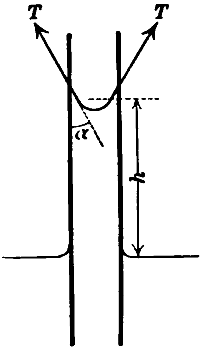


FIG. 108.

The rise of liquid in a cylindrical tube may be calculated as follows:

Let T denote the surface tension (Fig. 108),
r " " radius of the tube,
h " " mean hight of the liquid,
ρ " " density of the liquid,
α " " angle of contact;

then the total upward force in the direction of the axis is $2\pi r \cdot T \cos \alpha$, and the weight of the column of the liquid is $\pi r^2 h \rho g$. Since these forces are in equilibrium,

$$h = \frac{2T \cos \alpha}{\rho r g}. \tag{5}$$

If $\alpha > \frac{\pi}{2}$, h becomes negative, or the surface is depressed. The fact expressed by this equation, namely, that the elevation is inversely as the diameter of the tube, was one of the first observed laws of capillarity.

The calculation of the rise of a liquid between parallel plates is quite similar to that for tubes.

If l be the length of the plates measured parallel to the

liquid surface, and d the distance between them, the total upward force will be $2lT \cos \alpha$, and the weight of the elevated column $hld\rho g$; whence

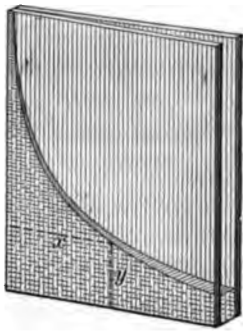


FIG. 109.

(6) $$h = \frac{2T \cos \alpha}{d\rho g}$$

When two glass plates are moistened with water and then brought together along a vertical edge, as in Fig. 109, the liquid appears bounded by a curve whose equation may be thus found: Let y denote the ordinate of any point in the curve, let x be the distance of the point from the common edge, and d the distance between the plates at the point xy. Then by geometry $d \propto x$; also, by the law already demonstrated for parallel plates, $y \propto \frac{1}{d}$; whence

$$x \propto \frac{1}{y},$$

or,

(7) $$xy = \text{constant},$$

which is the equation of the equilateral hyperbola.

When a liquid drop is placed in a conical tube, which it wets (Fig. 110), the drop will move toward the small end of the tube, for the reason that the area of the free surfaces df and eg is diminished, *i.e.* the potential energy of the drop is less in the final than in the initial position. For an analogous reason, a drop placed in a tube which it does not wet will move toward the larger end.

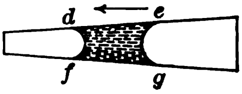


FIG. 110.

110. Determination of Surface Tension. — An approximate value of the surface tension of a liquid may be found from equation 5 when the angle of contact is known.

Quincke has devised a method by which the surface tension may be found from measurements of a drop lying on a plate, as in Fig. 104, T being calculable as a function of the hight k of the point at which the tangent to the curve becomes vertical, the density of the liquid, and the acceleration of weight.

The following table shows Quincke's results for a few common liquids:

TABLE OF SURFACE TENSIONS.

Liquid.	Tension in Dynes per Centimeter, at 20° C. of Surface separating the Liquid from		
	Air.	Water.	Mercury.
Water . . .	81	0	418
Mercury . .	540	418	0
Alcohol . . .	25.5	—	399
Olive Oil . .	36.9	20.6	335
Turpentine . .	29.7	11.6	250

The most accurate method for the determination of surface tension is one proposed by Tait, in which observations are made on the velocity of propagation of ripples on the surfaces of the liquid (see Art. 477).

111. Experiments on Soap Films. — In order that the form of a soap film be stable, it is necessary that there shall be a constant pressure within and a constant pressure without; for, since the substances concerned are fluids, if there were an excess of pressure at one point over another, an alteration in the arrangement of parts would ensue until the difference of pressure was reduced to zero.

In other words, the difference of pressure between the outside and the inside of the film must be constant, that is,

$$p = T\left(\frac{1}{r} + \frac{1}{r'}\right) = \text{constant};$$

or, finally,

$$\frac{1}{r} + \frac{1}{r'} = \text{constant}. \tag{8}$$

That is to say, the mean curvature at every point of the film must be the same.

The stable forms of soap films which are also surfaces of revolution are six in number. If r denotes the radius of curvature of the meridian section at any point, and r' the radius of curvature in a normal section at right angles to it, the analytical conditions for these surfaces may be classified as follows, using the notation $|r|$ to denote the numerical value of the radius:

Case.	r	r'	p	Name.
(1)	$+$	r	$+$	Sphere
(2)	$+$	$<r$	$+$	Unduloid
(3)	$+$	$>r$	$+$	Nodoid
(4)	∞	$+$	$+$	Cylinder
(5)	∞	∞	0	Plane
(6)	$-$	$<\|r\|$	$+$	Unduloid
(7)	$-$	$-r$	0	Catenoid
(8)	$-$	$>\|r\|$	$-$	Nodoid

Case 1. The condition that r shall be equal to r' determines a sphere, the pressure in this case being greater within than without. This surface is readily formed by blowing a bubble and detaching it from the pipe by a sudden motion of the hand.

Case 2. The surface in this case is one form of the figure known as the *unduloid*. Both curvatures are positive

and the pressure greater within than without. It may be formed by securing a soap bubble upon two circular rings, and then separating them a certain amount, as in Fig. 111. The ends will be observed to be closed by spherical caps.

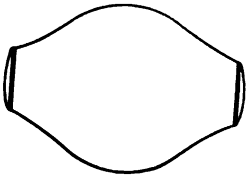


FIG. 111.

Case 3. When both curvatures are positive, and the curvature of the meridian section the greater, the surface is a form of the nodoid for which the pressure is greater within than without. It may be formed by squeezing a soap bubble between two discs, Fig. 112.

Case 4. When the curvature of the meridian section is reduced to zero, the surface is a cylinder. This surface may be formed by separating the rings an amount somewhat greater than for the experiment of Fig. 111, provided the bubble is not too large. When the length of the cylinder approaches the diameter, the figure becomes unstable. The pressure within is greater than that without, as is shown by the spherical caps. The radius of the caps is twice that of the cylinder.

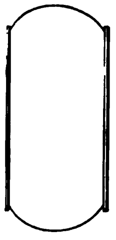


FIG. 112.

Case 6. The conditions here determine another form, or rather portion of the unduloid for which the meridian section has a negative curvature, and the perpendicular normal section a positive but greater curvature. It may be formed by separating the rings in Fig. 111 further than was required for the cylinder. If, however, this separation (Fig. 113) be carried too far, the surface will divide into two bubbles by spontaneous constriction. The forms of the unduloid are of interest as showing how a vein of water

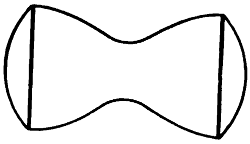


FIG. 113.

breaks up into drops, as is shown in Fig. 114. After the drops break away from the vein, they vibrate through a spherical form, becoming successively prolate and oblate spheroids.

Fig. 114.

Case 7. When the curvature of the meridian section is negative, and the perpendicular normal section has a positive and numerically equal value, the surface is known as the catenoid. It may be formed by joining two rings wet with soap solution, breaking the films across the circles and separating the rings,

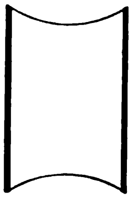


Fig. 115.

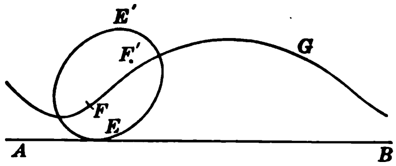


Fig. 116.

as in Fig. 115. The pressure is then clearly the same on both sides of the surface.

Case 8. The surface determined by these conditions has a form similar to that of Fig. 115, but differs from it in having the negative portion of the mean curvature numerically greater. As the pressure within is less than without, this form is not conveniently produced in a soap film.

The meridian curves of these six surfaces of revolution have the singular property of being in each case the roulette of the focus of a conic section, produced by rolling the conic on a straight line. If, for example, an ellipse, EE' (Fig. 116), be rolled on the line AB, the trace of the focus would be the undulating line FG, which is the generatrix of the unduloid. When the

foci of the ellipse coincide, the conic becomes a circle, and the roulette a straight line, or the generatrix of the cylinder.

When the eccentricity of the ellipse becomes unity by the coincidence of the foci with the extremity of the major axis, the roulette becomes a semicircle, which is the generatrix of the sphere.

A plane may be regarded as a portion of a sphere of infinite radius.

In the case of the parabola the roulette is the catenary, or meridian curve of the catenoid.

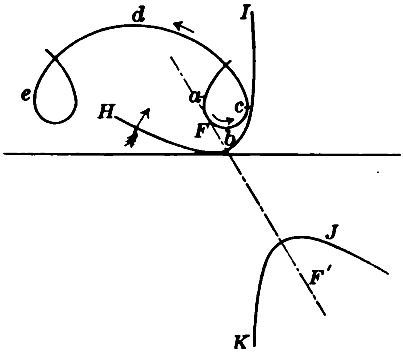


FIG. 117.

By rolling a hyperbola (Fig. 117) first on the branch *HI* and then on the branch *KJ*, the focus *F* will describe the

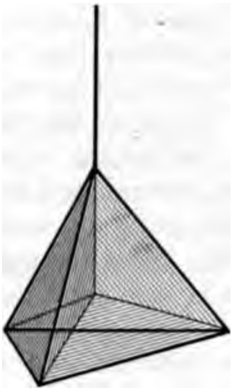


FIG. 118.

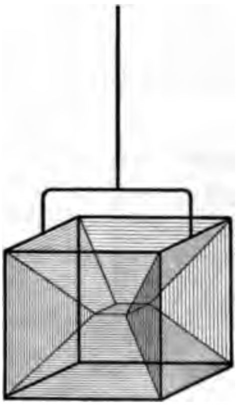


FIG. 119.

curve *abc* · · · *cde*, which is the meridian section of the nodoid, the portion *cde* corresponding to case 3 and *abc* to case 8.

A great variety of figures may be formed by dipping wire frames into a soap solution. Figs. 118 and 119 show two such, in which the bounding surfaces are all planes.

Whatever may be the form of the surface, if the pressure be the same on both sides, the curvature in one normal section will always be equal and opposite to that on the other. One familiar surface which has this property is the warped helicoid, which may be readily produced as a film on a wire bent into the form of a helix about a second wire as an axis.

Whenever three liquid surfaces meet upon a line, the angle between them must be 120°, since three equal forces in equilibrium when plotted as vectors form an equilateral triangle.

112. Limit of Molecular Range. — In order to obtain an estimate of the distance at which the forces concerned in the phenomenon of surface tension become inoperative, Quincke made use of a glass plate covered with a wedge-shaped layer of silver. A glass rod was first laid on the plate, and the latter covered with a silver solution, from which the silver was deposited in a uniform layer except under the rod, where the thickness gradually fell away. The metal was then removed from one-half of the plate, the surfaces carefully cleaned, and the apparatus partly immersed in water, as shown in Fig. 120. Since the angle of contact for glass is small, and that for silver is about 90°, the water will rise on the glass surface above the level on the silver plate. But since the film is continuous, there will be near the edge of the silver a region, BC, where the line of contact curves upward to meet that on the glass. The thick-

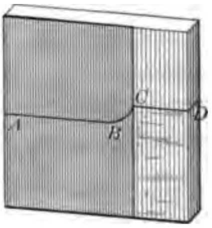


FIG. 120.

ness of the metal at the point B, where the molecules of the glass begin to make their influence felt through the layer of silver, will be the limit of molecular range. On converting the silver into a transparent salt, Quincke was able to estimate by optical methods that this distance was $\frac{1}{20000}$th of a millimeter, $5(10)^{-6}$ cm., or $\frac{1}{500000}$th of an inch; that is to say, of a tenth of the magnitude of the wave-length of light. The colors exhibited by soap films furnish another method for estimating the range of these molecular forces. As the film gradually grows thinner, the color passes through the series of hues red, yellow, green, blue, and black, after which the film soon bursts. As this phenomenon is due to a direct relation between the thickness of the film and the wave-length of light, it furnishes a basis for the estimate that the minimum thickness which a soap film can attain is $1.2(10)^{-6}$ cm. Down to this degree of thinness there is no observed falling off in the contractile force.

The principles of thermodynamics show that it is not possible that this thickness can be reduced to one-hundredth of this amount without a considerable diminution of the surface tension; for if this did not occur the energy which it would be necessary to supply to the film to reduce it to 10^{-8} cm. would be more than enough to convert the liquid into vapor at atmospheric pressure. Hence, since this diminution could occur only when the film was reduced to a layer of a few molecules and a thickness well within the molecular range, the conclusion may be drawn that the thickness of the black spot is of the same order of magnitude as the range of the molecular forces.

113. Formation and Growth of Raindrops. — The influence of surface tension on the formation and growth of raindrops was first pointed out by Lord Kelvin, who based his

reasoning on the following experiment. Let C (Fig. 121) represent a vessel exhausted of air and partly filled with a liquid in which is placed a capillary tube, B. The space above the liquid will thus contain only saturated vapor.

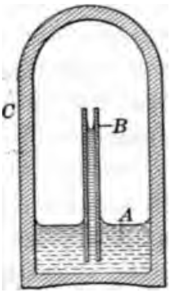


Fig. 121.

Now, when equilibrium has been established within C, that is to say, when the evaporation and condensation at either of the surfaces A or B is the same, it is evident that the vapor pressure at the level A must exceed that at B by the weight of a column of the vapor of unit cross section and hight BA. But the curvature at B, being negative, is less than at A, where it is zero; whence the conclusion may be drawn that when a vapor is in equilibrium with its liquid, the curvature of the surface must be greater where the pressure is greater.

Since this pressure would have to be very great if the radius of curvature of a drop were very small, it is difficult to see how raindrops could begin to form, unless upon some nucleus, such as a speck of dust or a group of air particles. It is thus easy to see that a rainfall must clarify the atmosphere, by removing from it the small particles of foreign matter. According to Kelvin's principle, the increase in size of raindrops will occur, not by the union of two or more drops which are jostled together, but by the growth of the large ones at the expense of the small ones. Thus, suppose that D and d (Fig. 122) are two drops of unequal size in a cloud. Then if the vapor pressure in the cloud is just right

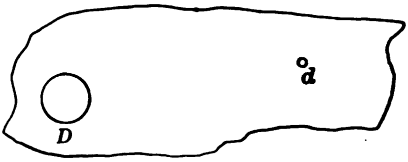


Fig. 122.

to preserve equilibrium at the surface of D, which has the smaller curvature, it will be too small for d, which will evaporate, increasing the pressure in the cloud and producing condensation at the surface of D. Similarly, if the pressure were initially right for d, it would be too large for D, so that condensation would occur, resulting in a diminution of the vapor pressure in the cloud and evaporation from d. Thus, it appears in either case that the large drops grow at the expense of the small ones.

114. Surface Tension and Small Floating Bodies. — If a needle be carefully placed on the surface of still water, it may often be observed to float, although the density of steel is more than seven times that of water. The relation of the forces by which this result is effected may be seen in Fig. 123. Since the angle of contact between steel and water does not vary much from 90°, as may be judged by the hemispherical form which a small drop assumes when placed on a knife blade, the surface tension furnishes two forces whose vertical components, in the case under consideration, are equal to the weight of the needle diminished by the weight of the water displaced.

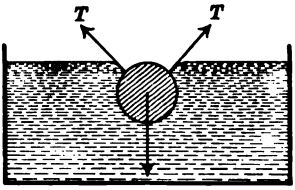


FIG. 123.

The insects, often seen on the surface of pools, derive their support in a similar manner.

The tendency often exhibited by small floating bodies on the surface of a liquid to cling to the sides of the vessel, or to collect in groups, is also a phenomenon of surface tension and may be thus explained.

Suppose A and B (Fig. 124) are two plates suspended near together in a liquid which wets them, then it is obvious

that the curvature of the surface at m will be numerically greater than at l or n. But this condition, according to the law demonstrated in Art. 111, requires a smaller pressure just below m than that above, which is sensibly the same as that

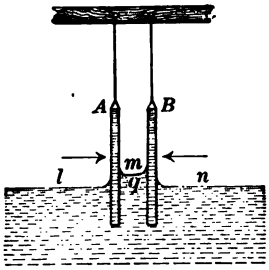


FIG. 124.

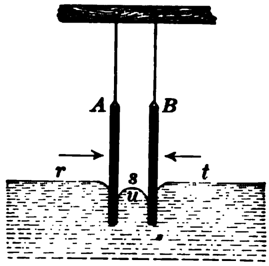


FIG. 125.

at l or n. The difference of pressure between q and l or n thus gives rise to a resultant force which urges the bodies into contact.

If the liquid does not wet the plates, the curvature at s (Fig. 125) requires that the pressure at u shall be greater than at s, which is the same as at r or t. This condition being met by a depression of the surface at s, the bodies are urged together by the excess of the pressure at this level over the pressure at s.

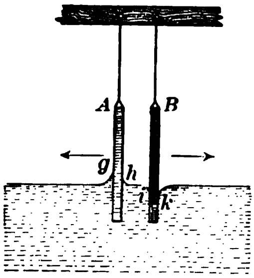


FIG. 126.

In the case where one of two plates is wet and the other not, it may be shown that they would be forced apart. Thus, if A (Fig. 126) is the plate which is wet, the liquid will rise higher at g than at h, and be depressed lower at k than at i. From the curvature of the surfaces it follows that the pressure to the left of gh

is less than that to the right, *i.e.* atmospheric pressure, while for similar reasons the pressure to the left of *ik* is greater than that to the right of this region. Hence, the result of these unbalanced forces will be a separation of the plates.

115. Phenomena Depending on Difference of Surface Tension. — If a film of water lying on a plate be touched with a glass rod moistened with alcohol or ether, or if the rod be only approached very close to it, the water will be seen to draw back on all sides, leaving a spot entirely bare. This is due to the reduction of the surface tension where the water is mixed with the alcohol or ether.

The phenomenon known as tears of strong wine, to which Proverbs xxiii: 31 is possibly a reference, has a similar explanation. When the sides of a glass containing some strong wine are moistened by shaking the liquid, the film which adheres to the glass will be seen to draw together into little ridges which slide down in fairly large drops. Just as they reach the surface, however, they may be observed to stop or even shrink back. The ridges were formed initially in those portions of the film from which the alcohol had evaporated more rapidly and the surface tension was consequently greater. On reaching the surface of the liquor the alcohol vapor reduced the surface tension sufficiently to permit the stronger tension at the upper side to draw the drop back.

A fragment of camphor gum dropped on the surface of water is usually set gyrating in a singular manner under the action of the forces in the surface, which vary from point to point as the camphor dissolves.

116. Behavior of Oil on Water. — When a drop of a mobile oil is placed on the surface of water, it will spread indefi-

nitely, or until the layer is so thin that the force of surface tension is reduced, provided the temperature of the water is sufficiently low. If, however, the water is heated, the oil may be observed to gather together in drops. The explanation is as follows. Let Fig. 127 represent a drop of oil or water, and denote the tensions of the three surfaces by $_aT_w$, $_aT_o$, and $_oT_w$. Now, when the water is cold, the tension in the water-air surface exceeds the sum of the horizontal components of the tensions in the oil-air and the oil-water surfaces; but when the temperature is raised, although the tension in all the surfaces is diminished, the value of $_aT_w$ falls off most rapidly, so that equilibrium is ultimately established between the forces at the edge, and the oil assumes a lenticular form.

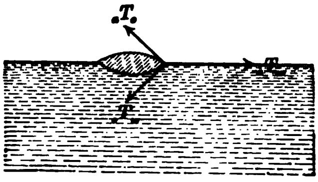


FIG. 127.

A curious effect is obtained by blowing on the surface of hot soup or chocolate on which there are a number of floating drops of oil. This is due to sudden changes in the relative surface tensions, consequent on the alteration of temperature.

117. Surface Tension as Related to Cleanness of Surfaces. — The high value of the surface tension of mercury renders it exceedingly difficult to preserve a clean surface of this liquid, and the same thing is true for water, for which the surface tension, although smaller than for mercury, still considerably exceeds that of any other common liquid. In the case of water, the high value of the surface tension must be regarded rather as an advantage than a disadvantage, since this property renders it just so much more valuable as a cleanser.

In using benzine to remove grease spots the considerations already presented show that the application of the benzine to the center of the spot would naturally result in driving the grease into the clean parts of the cloth. If, however, the benzine be applied in a ring about the spot, the oil might be driven in toward the center, whence it could be absorbed by some porous substance. The diminution of the surface tension with the temperature may be utilized in the removal of grease spots by covering one side of the cloth with blotting paper or fuller's earth, and applying a hot flatiron to the opposite side.

EXAMPLES.

1. How high will water stand above hydrostatic level in a tube 0.057 cm. in diameter, assuming the angle of contact to be very small?

Ans. 5.8 cm.

2. How much will the level of mercury be depressed in a glass tube 0.067 cm. in diameter, the surface tension being taken as 540 dynes per cm. and the angle of contact as 135°? *Ans.* 1.71 cm.

3. What is the pressure within a soap bubble 12.5 cm. in diameter if the tension of the liquid be taken as 80 dynes per cm., and what is the total pressure exerted by the film on the gas within?

$p = 51$ dynes / cm^2; total pressure $= 25{,}100$ dynes.

4. A horizontal disc of radius 14 cm. is held up by means of a film of water 0.21 cm. thick between it and a similar disc. Assuming that the surface of the water at the edge has a radius of curvature in a meridian plane equal to half the distance between the plates, find the weight of the lower disc and the water. *Ans.* $4.7(10)^5$ dynes.

PART II. — HEAT.

CHAPTER VIII.

THERMOMETRY.

118. Temperature. — Two bodies are said to be at the same temperature if, on being brought in contact, neither body grows warmer. If when two bodies are brought in contact one grows warmer and the other colder, the body which becomes colder is said to be at a higher temperature than the other. This difference of temperature will be very nearly proportional to the rate at which this change goes on.

To effect the change there must be a transfer from the hotter to the colder body of a definite and measurable physical magnitude, called *heat*. No hypothesis as to its ultimate nature need be made for the present.

119. Scale of Temperature by Mixtures. — Experiment shows that, for a given pressure, the change of a substance from one state to another takes place at a constant temperature. Accordingly, in order to form a scale of temperature, the temperature of melting ice under a pressure of one atmosphere may be taken arbitrarily as zero, and the temperature of boiling water at the same pressure as 100. The temperature of the mixture formed by taking equal parts of boiling and freezing water might then be called 50, or, in general, the temperature of any mixture of different amounts

of water in the standard states might be taken as numerically equal to the percentage of hot water used to form it.

The temperature of any body on this arbitrary scale could then be found with no other thermoscope than the temperature sense. Thus, the temperature of a body which neither grew warmer nor cooler when immersed in a mixture of three parts boiling water and one part freezing water would have a temperature of 75.

The scale formed by dividing the difference of temperature between freezing and boiling water under the standard conditions into 100 equal steps or degrees is called the Centigrade scale.

The arbitrary scale of mixtures just described would coincide essentially with the Centigrade scale, but in practice the change of volume which accompanies the change in temperature of a body affords a much more convenient measure of difference of temperature.

120. The Temperature Sense. — The temperature sense supplies quite a delicate test of equality of temperatures, provided that they are not widely different from that of the body, in portions of the same kind of matter, the range of error being about $\frac{2}{10}^\circ$ C. However, when comparison is made between different substances, the impressions received through it cannot be relied upon. Thus, if on a frosty day in winter one touch in succession several different objects, known to be of the same temperature, a piece of metal will feel very cold, a piece of wood much less so, while a woolen cloth may even seem warm. The sensation depends on the rate at which heat is lost, and this rate is a function not only of the difference of temperature between the substance and the hand, but also of the nature of the material. Since a metal conveys the heat away more rapidly than wood or wool,

it feels colder. So, again, if wool conducts the heat of the body away less rapidly than the air of the room, the cloth will feel warm by contrast.

If the right hand be held a few minutes in a cold bath and the left in a hot one, and both be then plunged into tepid water, the water will feel warm to the right hand but cold to the left, for the reason that the surface of the former is receiving heat, while that of the latter is losing it.

121. Expansion. — It may be stated as a general law that the volume of bodies increases continuously with the

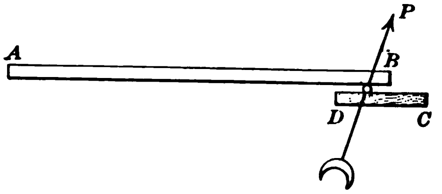


FIG. 128.

temperature. The few exceptions are confined to small ranges of temperature.

The expansion of solids may be illustrated by the apparatus shown in Fig. 128. *AB* is a metal bar fixed at *A* and resting at *B* on a minute cylinder, which is free to roll on the plate *DC*, and carries a pointer, *P*. If the temperature of the bar *AB* be raised by a flame applied at different points, the motion of the end *B* toward the right will be indicated by the revolution of the pointer.

To show the expansion of liquids, let a tube of small bore with a bulb blown at the bottom, as in Fig. 129, be filled with any liquid to a point, *a*. If the temperature be gradually raised by immersing the bulb in a vessel of hot water, the top of the column will be observed to rise from *a* to

some point, *b*. The observed expansion in this case is really a differential effect, or the difference between the dilatation of the liquid and that of the containing vessel.

The expansion of gases may be shown by inverting the tube filled with air in a beaker of inky water, as in Fig. 130. On warming the bulb the expanding gas will escape in bubbles from the bottom of the tube.

If the source of heat be withdrawn, the air still remaining in the apparatus will contract and the inky water rise a considerable distance in the tube.

Under proper precautions any one of these experiments might be made to yield numerical values for the change of temperature experienced by the body. The apparatus when so arranged would be called a *thermometer.*

FIG. 129.

122. Choice of the Thermometric Substance. — The choice of the thermometric substance should depend on its fulfillment of the following conditions: 1°. The substance must return accurately to the same volume on being brought to its initial temperature. 2°. It must admit of a considerable change of temperature without change of state.

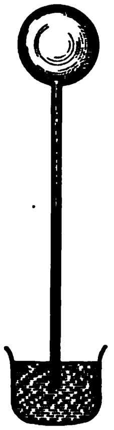


FIG. 130.

A gas answers these requirements perfectly, and furnishes the most accurate measure of differences of temperature attainable, but its use is subject to the inconvenience of change of volume under change of pressure, as well as of temperature.

Among the liquids, mercury fulfills the conditions most nearly. Its law of expansion over a wide range, say from $-40°$ C. to $330°$ C., about the temperature of melting lead, is sensibly the same as that

of air. The discrepancy between a mercurial and an air thermometer does not, from 0° to 100°, exceed 0.03° C. Moreover, mercury is easily obtained, is very opaque, and hence easily seen, and does not cling to the glass. Liquids which wet glass, such as alcohol, ether, etc., are occasionally employed for moderate and for low temperatures where mercury would freeze. They offer the advantage of a much greater expansion for a given difference of temperature, and a smaller density permits the use of larger bulbs without increasing the liability to breakage, or disturbance of the reading through distortion of the bulb under the weight of the liquid.

The use of volatile liquids necessitates keeping the bulb and stem at the same temperature, for otherwise vaporization into the upper portion might occur, and a false reading be given by the difference in temperature between the bulb and the stem. The top of such a liquid thread also is not as easily read as a mercury column, on account of its transparency.

123. Air Thermometer. — A convenient form of air thermometer devised by Jolly is represented in Fig. 131. A is a glass bulb filled with air and communicating with the tube BD through a small bore. EC and BD are two glass tubes connected by the flexible pipe FG and filled with mercury. By raising or lowering EC the level in BD may always be brought to a fixed point scratched on the glass.

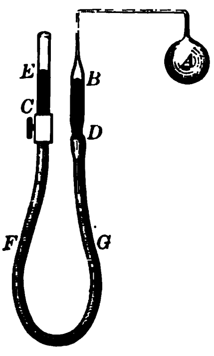


FIG. 131.

The difference between the levels E and B may be read from a graduated mirror placed behind the tubes. This difference

added to the barometric hight gives the pressure of the air in A. Call h_0 the pressure in the bulb at the temperature 0° C., and h the pressure at any other temperature, $t°$; then, by the laws of gases explained below (Art. 138),

$$h = h_0\left(1 + \frac{1}{273}t\right), \tag{1}$$

or

$$t = \left(\frac{h}{h_0} - 1\right) 273. \tag{2}$$

The greatest source of uncertainty in the use of the air thermometer is introduced by the expansion of the glass, but this may be corrected by careful investigation of the volume of the bulb throughout the range of temperature for which it is to be employed.

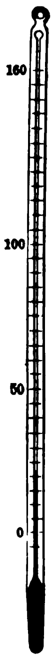


Fig. 132.

124. Mercurial Thermometer. — The thermometer best suited to a large variety of practical purposes consists of a glass tube with a capillary bore and ending in a bulb filled with mercury. A form much used in the laboratory is shown in Fig. 132. Three processes are necessary to prepare it for use.

1°. The filling: In order to introduce the mercury, the bulb is heated so as to expel a portion of the air, and the open end of the stem is immersed in a vessel of the liquid. On cooling the bulb some of the mercury is forced in by the pressure of the outside air. The liquid thus introduced is boiled till the air is all expelled and the instrument filled only with mercury and its vapor. If the end of the stem be again dipped into the vessel of mercury, the vapor will be condensed and the tube completely filled with the liquid. The whole instrument is then raised

to the highest temperature it is intended to register, and the tube hermetically sealed.

As the volume of glass is likely for a time to undergo sensible changes dependent on its previous history, the tube is laid away for a period of six months or more before it is marked.

2°. Determination of the fixed points: To determine the freezing point, the bulb and stem, as far as the top of the column of mercury, are immersed in moist pounded ice until the level of the mercury becomes stationary. The position of this point is then carefully marked by a scratch on the glass.

The boiling point is found by immersing the bulb and stem in the steam issuing from water boiling under a pressure of 760 mm. The stationary position of the top of the column is recorded as before by a scratch on the stem.

3°. The graduation: The interval between the points of reference is now divided into 100 parts of equal volume, if the instrument is to be a Centigrade thermometer, and the graduation continued as far beyond the 0° and 100° points as may be desired. If great accuracy is required, the errors of the graduations should be obtained by comparison with a standard instrument, or by some method of calibration.

125. Scales of Fahrenheit and Réaumur.—The Centigrade scale, first constructed by Celsius in 1742, is most used for the record of scientific observations, but for the operations of daily life a different scale, devised by Fahrenheit of Danzig, is more familiar to the English-speaking people. On this scale the freezing point is marked 32° F. and the boiling point 212° F. The reason for Fahrenheit's division is not certainly known. It has been conjectured that the interval between the boiling and the freezing point was

divided into 180 parts from the analogous division of a semicircle into 180 degrees. But as Fahrenheit is known to have constructed many thermometers previous to the discovery that water had a constant boiling point for a given pressure, it seems far more probable that he chose as the upper fixed point the temperature of the human body taken in the mouth or under the armpit. The lower or zero point of his scale was the coldest then known temperature, that of a mixture of snow and salt.

The interval between these points was first divided into 24 parts, and later into four times as many, *i.e.* 96 parts, a duodecimal division.

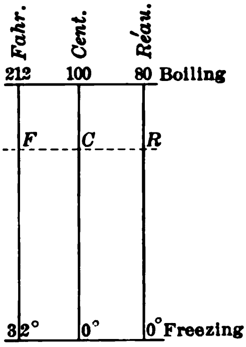


FIG. 133.

The advantages of Fahrenheit's scale are the convenient size of the degrees and the rare need of negative readings.

Another scale common in Germany and bearing the name of Réaumur is marked zero at the freezing point and eighty at the boiling point. It has nothing to commend it.

The relation between the readings of the three scales may be found thus:

Let F, C, and R be the respective readings for any temperature. Then, since the distance between the boiling and freezing points on each scale is divided in the same ratio by the given temperature,

$$(3) \qquad \frac{F-32}{180}=\frac{C}{100}=\frac{R}{80},$$

or,

$$(4) \qquad C=\frac{5}{9}(F-32)=\frac{5}{4}R.$$

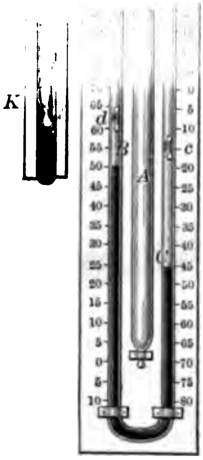


FIG. 134.

for the rise of the c
mercury in each arm
which is held at any
left by the friction c
side of the tube, as
view at *K*.

When the liquid in
the mercury rises in
the index before it.
perature falls, this inc
the highest point atta
time the index on the
will in a similar mann
temperature reached.
replaced by means of a

A self-registering minimum thermom
a small index of glass (Fig. 134 A) pla
of a fluid, such as alcohol, which wets
glass. The stem of this thermometer b
placed in an inclined position, so that
bulb is a little lower than the opposite
the liquid will flow past the

mometer, except that the bore is very much reduced at a point near the bulb.

When the temperature rises, the expansion of the mercury in the bulb forces a certain amount of the liquid past the constriction; but when the temperature falls again, the thread breaks off at this point, leaving all the column which has passed it in the tube. The instrument is set again by shaking the mercury down past the obstruction.

127. Weight Thermometer. — The apparent expansion of a liquid may be obtained by weighing the overflow occasioned by a rise of temperature in a vessel just full at zero. Such an instrument, called a weight thermometer, is represented in Fig. 135.

FIG. 135.

It consists of a bulb with a bent capillary stem open at the end. Let M_0 be the mass of a liquid, say mercury, necessary to fill it at zero, and m the overflow when the temperature is raised from 0° to t°. Then, assuming the increase in volume, v, of a liquid to be proportional to the rise of temperature, and calling the initial volume V_0,

$$v = aV_0t, \tag{5}$$

where a is some constant. But the volumes occupied by the masses at the temperature t are proportional to these masses; whence

$$\frac{v}{V_0} = \frac{m}{M_0 - m}, \tag{6}$$

which gives

$$m = a(M_0 - m)t, \tag{7}$$

or, solving for t,

$$t = \frac{m}{a\,(M_o - m)}, \tag{8}$$

where a may be determined once for all by an observation at 100° C., thus,

$$a = \frac{m_{100°}}{100\,(M - m_{100°})}. \tag{9}$$

128. Other Thermometers. — The pressure-gauge of Bourdon, described in Art. 95, may be made to serve as a thermometer by filling the tube with glycerine or alcohol. Expansion or contraction of the liquid will then actuate the pointer in the same manner as would the variations in the pressure of a gas.

A thermometer devised by Bréguet consists of a ribbon made of three thin strips of platinum, gold, and silver, fastened together in the order named, and coiled into a helix so that the silver shall be on the inner face. The unequal expansion of these metals causes the helix to unwind, carrying a pointer over a properly graduated scale.

Two methods, founded on the electrical properties of bodies, have been extensively used for estimating differences of temperature. One of these depends on the fact that if one junction in a conducting circuit, formed of dissimilar bodies, is heated or cooled, a current of electricity will be produced. The other method, of great value in the estimation of both high and low temperatures, is based on the variation of the electrical resistance of metals with change of temperature. Detailed description of these instruments may be more properly presented in the chapters on electricity.

EXAMPLES.

1. What is the temperature 92° Fahrenheit on the Centigrade scale? *Ans.* 33.3° C.

2. What temperature is expressed by the same number on the Centigrade and Fahrenheit scales? *Ans.* $-40°$ C.

3. If the hight of the column of mercury which measures the pressure of the air in an air-thermometer at 0° is 78.4 cm., what is the temperature when this hight is increased to 96.3 cm.? *Ans.* 62.3° C.

4. A weight thermometer weighs 104.5 gms. empty, and contains 623.5 gms. of mercury at 0° C. At what temperature, assuming coefficient of expansion as 0.000154, will it weigh 717.6 gms.? *Ans.* 110° C.

5. If a weight thermometer which contains 324 gms. at zero loses 4.97 gms. by an increase of temperature of 11.9°, what is the coefficient of relative expansion? *Ans.* 0.00131.

CHAPTER IX.

EXPANSION.

129. Coefficient of Expansion. — Let the measure of any dimension of a body at 0° be called S_o, and its value at the temperature t under the same pressure be S, then the results of experiment may be expressed with sufficient accuracy by the linear equation

$$S = S_o(1 + \phi t), \tag{1}$$

where ϕ is a constant depending only on the nature of the material, and is known as the *coefficient of expansion.*

Solving for ϕ,

$$\phi = \frac{\frac{S - S_o}{t}}{S_o} = \frac{\text{increase per degree}}{\text{size at zero}}, \tag{2}$$

or the coefficient of expansion may be defined as the ratio of the increase per degree to the size at zero.

130. Expansion of Solids. — In treating of the expansion of solid bodies, three coefficients are commonly considered, according as S is regarded as a length, an area, or a volume.

The equation 1 becomes in each case

$$L = L_o(1 + \lambda t), \tag{3}$$

$$A = A_o(1 + \sigma t), \tag{4}$$

$$V = V_o(1 + a t). \tag{5}$$

λ, σ, and a are called, respectively, the coefficients of linear, superficial, and cubical expansion.

To find a relation between these coefficients, choose three directions at right angles in the body, supposed isotropic, and let them be denoted by accents. Then,

$$
\begin{aligned}
L' &= L_o' \ (1 + \lambda t), \\
(6) \qquad L'' &= L_o'' \ (1 + \lambda t), \\
L''' &= L_o''' \ (1 + \lambda t),
\end{aligned}
$$

by multiplication,

$$L' L'' L''' = L_o' L_o'' L_o''' (1 + \lambda t)^3,$$

or,

$$V = V_o (1 + 3\lambda t + 3\lambda^2 t^2 + \lambda^3 t^3);$$

but as λ is a number of the order of $\frac{1}{100000}$, λ^2 and λ^3 may be ignored without sensible error, whence

$$(7) \qquad \alpha = 3\lambda.$$

In a similar manner it may be shown that

$$(8) \qquad \sigma = 2\lambda.$$

131. Determination of the Coefficient of Linear Expansion. —The linear coefficient of expansion may be satisfactorily

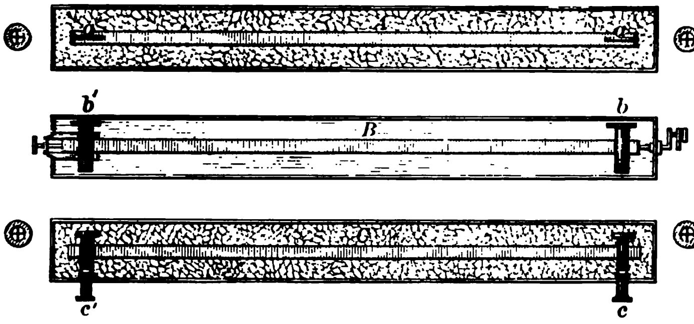


FIG. 136.

determined by the following method devised by Ramsden. The bar B (Fig. 136) to be investigated is immersed in a bath whose temperature can be accurately determined. On

each side are placed two iron bars, A and C, packed in ice, and intended to furnish a fixed length for purposes of comparison. At each end of C is a mounted eye-piece, c, c', with lens and cross wire. The bar B carries a fixed lens, b, at one end, and a similar lens, b', adjustable by means of a micrometer screw, at the other end. The bar A is furnished with a ring and cross wire at each extremity, as is shown at a, a'. The lenses on B and C are so designed that they form images of the cross wires at a and a'.

In order to make an observation, the image of the cross wire at a is brought into coincidence with the wire at c and the end b fixed; then after a', b', c' have been brought to a similar coincidence, with the temperature at zero, the bath about B is raised to some convenient temperature, t. A new coincidence of the cross wires is then found by shifting the lens b' a determinate distance, which is equal to the expansion of the bar $L-L_o$.

The coefficient of linear expression may then be calculated from

(9) $$\lambda=\frac{L-L_o}{L_o t}.$$

COEFFICIENTS OF LINEAR EXPANSION FOR SOLIDS.

SUBSTANCE.	MEAN EXPANSION PER DEGREE C.		
Brass	0.0000178	to	0.0000193
Cast Iron . . .	0.0000112		
Copper	0.0000168	to	0.0000188
Glass	0.0000083	"	0.0000120
Gold	0.0000140	"	0.0000147
Iron	0.0000116	"	0.0000119
Lead	0.0000280	"	0.0000292
Platinum . . .	0.0000068	"	0.0000092
Silver	0.0000191	"	0.0000212
Steel	0.0000103	"	0.0000132
Tin	0.0000223	"	0.0000229
Zinc	0.0000220	"	0.0000298

132. Expansion of Liquids. — In the case of fluids, the cubical coefficient is alone important; but it is customary to distinguish two values of this coefficient, the *absolute*, defined by equation 5, and the *apparent*, or that which is observed when the fluid is contained in a vessel which serves as a measuring flask.

The absolute expansion of a liquid may be obtained by the method of equilibrating columns adopted by Dulong and Petit in a very careful determination of this constant for mercury. The apparatus consisted of two tubes, A, C, connected by a portion of much smaller bore, and bent as $EBDF$, in Fig. 137.

The arm AB was surrounded with broken ice, and CD with a bath of any desired temperature. Let ρ_0, ρ be the densities of the mercury in the respective branches, and h_0, h the corresponding hights of the surfaces a, b above the axis of the tube BD.

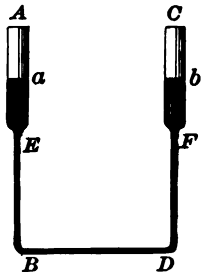


Fig. 137.

Then, since the pressure is the same at B and D,

$$h_0\rho_0 g = h\rho g, \tag{10}$$

also for one and the same mass of the liquid,

$$V_0\rho_0 = V\rho. \tag{11}$$

Combining (10) and (11) and substituting in (5),

$$h = h_0 (1 + at), \tag{12}$$

from which a may be found, since t is known.

The results of Dulong and Petit showed that the coefficient of expansion of mercury was sensibly constant between 0° and 100°, with a value

$$\frac{1}{5550}, \text{ or } 1.8018\,(10)^{-4} \text{ per degree.}$$

However, Regnault, using a similar method, found that this coefficient increased slightly with the temperature, but that the mean value from 0° to 100° was

$$1.8153(10)^{-4} \text{ per degree.}$$

To calculate the volume at any temperature, he gave the formula

$$V = V_o(1 + 0.000181792\,t + 0.000000000175\,t^2).$$

The expansion of other liquids cannot be satisfactorily expressed as a linear function of the temperature. In general, the coefficient increases with the temperature, becoming very large at high temperatures. The following table exhibits the deportment of some of the more familiar liquids.

EXPANSION OF CERTAIN LIQUIDS.

TEMP.	VOLUMES.			
	ALCOHOL.	ETHER.	BISULPHIDE OF CARBON.	OIL OF TURPENTINE.
0	1.	1.	1.	1.
10	1.01050	1.01518	1.01156	1.00919
20	1.02128	1.03122	1.02350	1.01875
30	1.03242	1.04829	1.03594	1.02865
40	1.04404	1.06654	1.04901	1.03886

133. Apparent Expansion. — Suppose a vessel graduated so that the reading of the scale divisions is true at zero. Let V_0 be the volume of any fluid at zero, and V_a the apparent volume at the temperature t, that is, the volume measured by this vessel.

These volumes are related by the equation

$$V_a = V_o(1 + at), \tag{13}$$

where $1 + at$ is the factor of apparent expansion. If V is the real volume, and α the true coefficient of expansion,

$$V = V_0(1 + \alpha t). \tag{14}$$

Considering only that portion of the vessel occupied by the liquid at the temperature t, it appears that this portion would have a capacity of V_a cubic centimeters if the glass were cooled to zero, for by supposition the graduation is correct for this temperature. Therefore, the volume of this portion of the vessel at the temperature t will be

$$V = V_a(1 + gt), \tag{15}$$

where g is the coefficient of expansion of the glass.

Eliminating the volumes between the three equations,

$$1 + \alpha t = (1 + at)(1 + gt), \tag{16}$$

or, approximately,

$$\alpha = a + g, \tag{17}$$

since the coefficients are small.

134. Expansion of Water. — Water for a short range presents a remarkable exception to the law of increase of volume with rise of temperature.

From 0° to about 4° it contracts, and afterwards expands in a manner exhibited by the table on the following page.

135. Maximum Density of Water. — The temperature of the maximum density may be roughly shown by the following experiment due to Hope. A tall beaker (Fig. 138), furnished at the top and the bottom with a thermometer, is filled with water and surrounded at the middle with a freezing mixture of ice and salt. At first the temperature of the lower thermometer will be observed to fall gradually, while the upper one is little affected, thus showing that the water descends as rapidly as it is cooled.

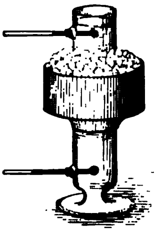


FIG. 138.

Temp. Cent.	Volume.	Temp. Cent.	Volume.	Temp. Cent.	Volume.
0°	1.000000	15°	1.000735	30°	1.00418
1	0.999948	16	890	35	1.0057
2	911	17	1.001057	40	76
3	889	18	235	45	97
4	883	19	424	50	1.0118
5	891	20	624	55	142
6	914	21	835	60	168
7	952	22	1.002057	65	195
8	1.000003	23	289	70	1.0225
9	068	24	530	75	256
10	147	25	78	80	288
11	239	26	1.00304	85	1.0321
12	344	27	31	90	356
13	462	28	59	95	392
14	593	29	88	100	1.0430

When the temperature of about 4° has been reached, the upper thermometer will begin to fall and continue to do so until it reaches zero. The lower one, however, remains constant at 4°, for the reason that the densest layer stays at the bottom, while the colder but lighter layers rise to the top.

Fig. 139.

A far more accurate method for determining the temperature of maximum density of water is that devised by Joule, in which he made use of two vertical cylinders about 15 cm. in diameter and 140 cm. high (Fig. 139), connected at the top by a channel, *a*, and at the bottom by a wide tube, *b*, with a stopcock. If the system be filled with water and the stopcock opened, the smallest difference in density between the two columns will produce a current in the channel, the presence of which may be indicated by a floating glass bead.

Joule proceeded to find a pair of temperatures, one above and one below the point of maximum density for which the density of water was exactly the same. By obtaining a series of such pairs of temperatures, for which the difference was smaller and smaller, the temperature of maximum density was found to be 3.95° C. within a small fraction of a degree.

One of the most recent determinations of the density of water is shown in the following table.

TABLE OF DENSITY OF WATER.

TEMP. C°.	DENSITY.	TEMP. C°.	DENSITY.	TEMP. C°.	DENSITY.
0°	0.999884	13°	0.999443	35°	0.99419
1	0.999941	14	0.999312	40	0.99236
2	0.999982	15	0.999173	45	0.99038
3	1.000004	16	0.999015	50	0.98821
4	1.000013	17	0.998854	55	0.98583
5	1.000003	18	0.998667	60	0.98339
6	0.999983	19	0.998473	65	0.98075
7	0.999946	20	0.998272	70	0.97795
8	0.999899	22	0.997839	75	0.97499
9	0.999837	24	0.997380	80	0.97195
10	0.999760	26	0.996879	85	0.96880
11	0.999668	28	0.996344	90	0.96557
12	0.999562	30	0.995778	100	0.95866

136. Expansion of Gases. — Experiment has shown that in the case of a gas under constant pressure, not only is the expansion strictly proportional to the increase of temperature, but that all gases have sensibly the same coefficient, namely,

$$\alpha = \frac{1}{273} = 0.00367.$$

This law, which may be expressed,

$$v = v_o\left(1 + \frac{1}{273}t\right), \text{ for } p \text{ constant,}$$

was discovered by Charles, and usually bears his name, though also termed by some writers the law of Gay-Lussac.

137. Boyle's Law. — The law by which a gas was defined in Art. 69, namely, that the product of the volume by the pressure is constant as long as the temperature remains unchanged, was first established by Robert Boyle, and published by him in 1662 in his "Defence of the Doctrine touching the Spring and Weight of Air."

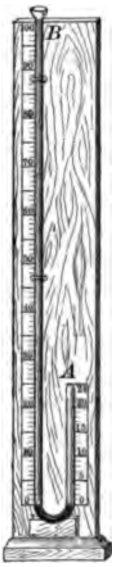


Fig. 140.

In his investigations, Boyle made use of a *U*-shaped tube of uniform bore, closed at the short end, as shown in Fig. 140. Mercury was first poured in till it stood at the same level in both legs, when the amount of air enclosed in the short branch was carefully noted. Small portions of mercury were then successively added, and the level of the columns recorded each time. The difference of these levels gave the excess of pressure of the enclosed air above that of the outside atmosphere. The volume could be read at once from the short leg.

If the initial pressure be called p_t and the corresponding volume v_t, and any other pressure and volume at the same temperature p_t', v_t', then the relation

$$p_t v_t = p_t' v_t' = k, \text{ a constant,} \tag{18}$$

is found to be true within the limit of errors of observation.

Equation 18 is known as Boyle's Law among all English-speaking peoples, and most others, except the French, who style it Mariotte's Law.

138. General Law for a Gas. — The laws of Boyle and Charles may be combined so as to yield an equation which is not subject to the condition of constant temperature or constant pressure. Thus, call the volume and pressure of a gas at zero respectively v_o and p_o. To find what the volume v will be if the temperature is raised to t and the pressure changed to p: Suppose, first, that the volume v_o is changed at the temperature zero until its pressure becomes p. Call this new volume v_o'.

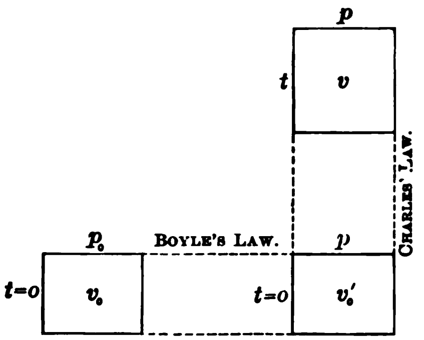


FIG. 141.

Then, since the temperature is constant,

$$p_o v_o = p v_o'. \tag{19}$$

Now let the temperature be raised to t, while the pressure remains constant. The final volume v is given by

$$v = v_o'\left(1 + \frac{1}{273}\,t\right). \tag{20}$$

Substituting the value of v_o' from (19) in (20),

$$v = \frac{p_o v_o}{p}\left(1 + \frac{1}{273}\,t\right), \tag{21}$$

which is the equation required. Clearing of fractions,

$$pv = p_o v_o\left(1 + \frac{1}{273}\,t\right), \tag{22}$$

which may be used to find any unknown quantity when the other four are given. Thus, it is frequently desired to know the volume at zero under a pressure of 760 mm. of mercury when the volume of a gas at a temperature, t, and a pressure, h, is v. Observing that the pressures are proportional to the hights of the column of fluids which measure them,

$$v_o = \frac{h}{760 \text{ mm.}} \frac{v}{1 + \frac{1}{273} t}. \tag{23}$$

Equation 22 may be simplified by writing

$$\tau = 273 + t, \tag{24}$$

when

$$pv = \frac{p_o v_o}{273} \tau = C\tau, \tag{25}$$

where C is some constant.

It is, however, obvious that if the volume and temperature of the gas remain constant the addition of more of the substance must increase the pressure, or C must contain, implicitly, the mass m, say,

$$C = mR,$$

so that, finally,

$$pv = mR\tau, \tag{26}$$

R being a constant which may be found from one observation. Thus, substituting the following values determined by Regnault for air,

$$v = 1 \text{ cc.},$$
$$m = 1.293(10)^{-3} \text{ gms.},$$
$$\tau = 273^\circ,$$
$$p = h\rho g,$$

where $h = 76$ cm.,

$$\rho = 13.60 \frac{\text{gms.}}{\text{cc.}},$$

$$g = 981 \frac{\text{cm.}}{\text{sec.}^2},$$

R is found to be

$$2.872\,(10)^6 \frac{\text{ergs}}{\text{gm. } 1^\circ \text{ C.}}.$$

139. Absolute Zero. — Assuming that the law,

$$pv = p_o v_o \left(1 + \frac{t}{273}\right),$$

is true for all temperatures,

$$pv = 0, \tag{27}$$

for $t = -273^\circ$ C.; that is to say, either the volume or the pressure would vanish at this temperature. The condition of no pressure may be explained on the kinetic theory of gases by supposing the molecules reduced to rest, that is, entirely deprived of heat. The alternative condition, namely, that the matter would occupy no volume, is hardly admissible.

The substitution of $\tau = t + 273$ in equation 22 is obviously equivalent to choosing a new zero, 273° C., below the freezing point of water. This point is called *absolute zero*, and the readings τ on this scale, absolute temperatures.

140. Compensated Pendulums. — The error in the running of a clock due to expansion or contraction of the pendulum rod may be satisfactorily compensated by the use of a mercurial bob proposed by Graham, and illustrated in Fig. 142.

FIG. 142.

To the extremity of the rod is attached one or more jars of glass or steel containing a considerable mass of mer-

cury. If a rise of temperature lengthen the rod, lowering the center of gravity, the expansion of the mercury, necessarily upwards, will produce the contrary effect. By the choice of a proper amount of mercury the effective length of the pendulum may be made to remain practically constant.

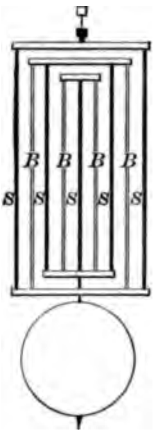


FIG. 143.

An equivalent design proposed by Harrison is shown in Fig. 143. From its form it is commonly known as the gridiron pendulum. The bars marked S are of steel. Their expansion will clearly lower the bob. Those marked B are of brass, and so arranged that increase of their length will raise the bob.

Let L be the combined length of the steel rods, and L' that of the brass ones, and call λ, λ' the corresponding coefficients of expansion. Then, in order that the length of the pendulum shall be invariable,

$$L\lambda t - L'\lambda' t = 0.$$

Whence

$$\frac{L}{L'} = \frac{\lambda'}{\lambda};$$

that is, the length of each material must be inversely as its coefficient of expansion.

141. Compensated Balance Wheel. — Fig. 144 shows the method adopted to compensate temperature changes in a chronometer. The rim of the balance wheel is constructed of two metals, the most expansible being placed on the outside and divided by saw cuts at a and a'. The effect of

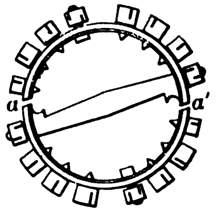


FIG. 144.

expansion on the rim is to throw the free end of each segment nearer the center, and thus correct the temperature changes in the elasticity of the hair-spring by the change in the moment of inertia of the balance wheel.

EXAMPLES.

1. A steel chain is 66 feet long at 25° C. What will be its length at 0°? *Ans.* 65 ft. 11.78 in.

2. A meter scale at 10° measures 99.981 cm. At 40° its length is 100.015 cm. What is the coefficient of expansion, and at what temperature is the scale correct? $\lambda = 0.0000113$; $t = 26.8°$.

3. What is the length of a brass wire which increases 1.231 cm. when heated through 200° C.? *Ans.* 342 cm.

4. One brass yard-measure is correct at 0° and another at 20°. What is the difference of length at the same temperature? *Ans.* 0.013 in.

5. If the iron bars in a compensated pendulum are 87.5 cm. long, what should be the length of the zinc bars? *Ans.* 39.9 cm.

6. A platinum and a zinc wire measure respectively 251.01 cm. and 249.97 cm. at zero. At what temperature will they have the same length, and what will this length be? $t = 243°$; $L = 251.55$ cm.

7. A cast-iron ball, 5.01 cm. in diameter at zero, rests upon a copper ring 5 cm. in diameter. To what temperature must both be raised in order that the ball shall just pass through the ring? *Ans.* 317° C.

8. What is the area of a brass plate at 80° which measures 20.32 cm. by 15.14 cm. at 0°? *Ans.* 308.5 sq. cm.

9. A piece of copper has a volume of 250.3 cc. at 0°. What will be its volume at 101°? *Ans.* 260.67 cc.

10. A glass flask contains 687 gms. of mercury at 70° C. What is its volume? *Ans.* 51.2 cc.

11. If the pressure of a gas is 8726 dynes per sq. cm. when its volume is 7375 cc., what will be its pressure at the same temperature if the volume is diminished to 1586 cc.? *Ans.* 40580 dynes / cm.^{2}.

12. What will be the volume of a gas at 382°, if the volume at 0° and the same pressure is 6580 cc.? *Ans.* 15780 cc.

13. If the pressure be unaltered, what will be the volume of a gas at 572° C. which occupies 2579 cc. at 198°? *Ans.* 4627 cc.

14. If the pressure of a gas in a vessel, whose expansion may be neglected, is 58.3 cm. of mercury at 98°, what will be its pressure at a temperature of 373°? *Ans.* 101.5 cm. Hg.

15. If the volume of a gas at zero is 2560 cc. under a pressure of 2.14 million dynes per sq. cm., what will be its volume at 95° under a pressure of 1.013 million dynes per sq. cm.? *Ans.* 7291 cc.

16. If 4490 cc. of a gas at 101°, under a pressure of 75.4 cm. of mercury, have its pressure increased to 82.1 cm. and its temperature raised to 225°, what will be the new volume? *Ans.* 5490 cc.

17. What will be the mass of a cubic meter of air at 50° under a pressure of 50 cm. of mercury? *Ans.* 719 gms.

18. If the volume of a gas is 22.1 cc. under a pressure of 70.8 cm. Hg. at a temperature of 16.5°, at what temperature will the volume be 19.4 cc. under a pressure of 76.0 cm. Hg.? *Ans.* 0° C.

19. Find the coefficient of expansion of a gas for the Fahrenheit scale and zero. *Ans.* $\frac{1}{459}$ per deg. F.

20. A tube 6.0 feet long, closed at one end and containing air, is half filled with mercury. If the open end is immersed in a vessel of mercury, how high will the mercury stand when the barometer reads 30 inches? *Ans.* 12 in.

21. A cylindrical diving bell 7.1 ft. high is lowered until the top is 20 ft. below the surface of the water. How far will the water rise within? *Ans.* 2.9 ft.

22. A glass tube used for sounding is 15.0 inches long and open at the lower end. The inside is covered with a soluble pigment, and the tube lowered to the bottom in sea water whose density is 1.03 gms. per cc. On raising to the surface it is found that the water had entered the tube a distance of 9.3 in. What was the depth of the water? *Ans.* 53.9 ft.

CHAPTER X.

CALORIMETRY.

142. Effects of Heat. — The effect produced by the application of a definite quantity of heat to a body is found to depend, in general, upon the nature and quantity of matter in the body, and its condition at the time the experiment is made. For example, let equal masses of alcohol, mercury, and ice be successively exposed for the same period of time to a Bunsen flame, which may for the present purpose be regarded as a constant source of heat. The resulting changes in temperature, volume, pressure, and state will be found to be markedly different in each body.

143. Unit of Heat. — Any of the effects resulting from the addition of heat to a given body, in a definite condition, might be used as the measure of a quantity of heat. The heat unit generally adopted is that quantity of heat which, added to 1 gm. of water at 0° C., will raise its temperature to 1° C. This unit is known as the *calorie*.

A unit one thousand times as great, and known as the *large calorie*, is often used in the investigations of Engineering.

144. Thermal Capacity. — When equal masses of unlike substances, at different temperatures, are kept in contact until they have assumed a common temperature, it is found, in general, that this resultant temperature is different from the average of the two, from which the inference may be drawn that different bodies require different amounts of heat to raise like masses through the same difference of tem-

perature. This conclusion really rests on four assumptions, which may be stated as follows: 1°. All the heat gained by one body is lost by the other, and *vice versa.* 2°. No action takes place between the bodies other than giving and receiving heat. 3°. If a body is made to pass through a series of states by the addition of heat, and is then allowed to cool, so as to pass through these same states in the reverse order, the quantity of heat which entered during the heating process is equal to that which left it during the cooling process. 4°. The effect of a given quantity of heat does not depend on the temperature of the source. Assumptions 1° and 2°, it will be observed, relate to certain conditions of the experiment which may be fulfilled with sufficient approximation. Numbers 3° and 4° are principles, by no means self-evident, whose truth has been established by experience.

The thermal capacity of a body is defined as the quotient of the heat received by the rise of temperature produced; that is, if Q units of heat raise the temperature of a body from t to t',

$$\text{thermal capacity} = \frac{Q}{t'-t}.$$

145. Specific Heat. — The specific heat of a substance may be defined as the thermal capacity per unit mass, that is, the heat per unit mass per unit rise of temperature. If Q units of heat raise m units of mass from t to t' degrees, then the specific heat is given by

$$s = \frac{Q}{m\,(t'-t)}. \tag{1}$$

The specific heat of substances varies a little with the temperature, but for most purposes it may be assumed as constant.

146. Specific Heat by Method of Mixtures. — Suppose that a mass, m_1, of a substance having a specific heat, s_1, and at a temperature, t_1, is mixed with a mass, m_2, whose specific heat is s_2, and whose temperature, t_2, is below t_1. If after an interchange of heat they assume a common temperature, t, then the heat lost by the first body will be

$$Q_1 = m_1 s_1 (t_1 - t), \tag{2}$$

and that gained by the second,

$$Q_2 = m_2 s_2 (t - t_2). \tag{3}$$

If no heat is received from or given to external bodies during the period of equalization,

$$m_1 s_1 (t_1 - t) = m_2 s_2 (t - t_2).$$

In practice m_2 is usually water, so that $s_2 = 1$, or

$$s_1 = \frac{m_2 (t - t_2)}{m_1 (t_1 - t)}. \tag{4}$$

147. The Water Calorimeter. — The form of apparatus employed for the determination of specific heats by the method of mixtures is shown in Fig. 145. The substance to be investigated is finely divided, placed in the wire basket F, and suspended in the tube AB which passes through the boiler M.

The liquid in this boiler is heated by a lamp underneath, and may be stirred by a plate, L, fastened to the rods r, r, passing through the cover. A thermometer, t, with the bulb inserted in the basket, serves to indicate the temperature as the heating progresses. The calorimeter proper, C, is a cylindrical vessel made of very thin brass, contained in a larger and stronger vessel, E, but thermally insulated from it by non-conducting supports. The inner vessel C contains a measured quantity of water at a known temper-

ature. To diminish loss by radiation the adjacent surfaces of both vessels are carefully polished. The calorimeter is mounted on rails so that it may be approached to *A*, have the basket dropped in, and then be quickly withdrawn. A wooden or cork partition, *PQ*, screens the calorimeter from the influence of the heating apparatus. After immersion of the heated substance in *C*, the water is continually stirred till the thermometer *T* indicates a stationary value of the temperature, which is then noted. The specific heat of the substance may now be calculated by the method of the preceding article.

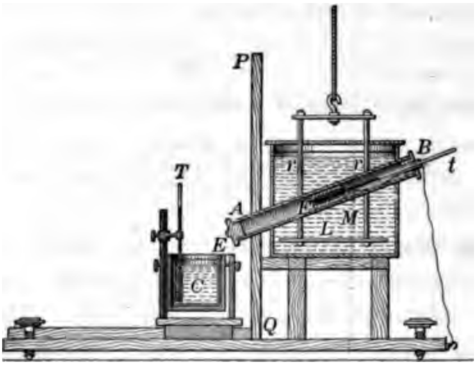


FIG. 145.

If, however, considerable precision is desired, correction must be made for the thermal capacity of the calorimeter, and for the gain or loss of heat by radiation. The former may be determined by a separate experiment. The latter may be compensated by the following procedure suggested by Count Rumford.

Let the highest temperature t reached by the calorimeter be found approximately from a preliminary experiment, and the temperature t_2 of the water in the calorimeter be chosen

so that it is as much below the temperature of the room as t is above it. Then while the temperature of the water is rising to that of the room the calorimeter will receive heat; but during the remainder of the process it will lose heat, since it is hotter than the temperature of the room.

148. Latent Heat. — When heat is continuously applied to a solid body, its temperature rises continuously, but only to a certain point, where it begins to melt.

If the mixture of solid and liquid be well stirred, further application of heat will melt more of the solid without affecting the temperature, until all of the substance has passed into the liquid state. The constant temperature at which fusion takes place is characteristic of each substance, and is called the *melting point.*

The amount of heat required to convert a gram of a solid at the melting point into a liquid at the same temperature is called the *latent heat of fusion.*

Quite analogous phenomena occur when a liquid under a definite pressure passes into the aeriform condition. The quantity of heat required to convert a gram of a liquid into a vapor without rise of temperature or change of pressure is called the *latent heat of vaporization.*

149. Ice Calorimeter. — The ice calorimeter is an instrument for measuring quantities of heat by means of the amount of ice melted, devised by Black, the discoverer of the phenomenon of latent heat. It consists of a block of pure ice, free from bubbles, in which a cavity is hollowed out and closed by a slab of ice.

To determine the specific heat of a substance a known mass of the body is raised to a convenient temperature and introduced into the cavity, which has previously been wiped

dry. The temperature of the body will quickly fall to zero, and some of the ice will be melted. This water is carefully collected by a cold sponge or a piece of blotting paper, and weighed. Suppose t was the temperature of the substance when introduced, and m its mass. Let s be the specific heat, L the latent heat, *i.e.* the amount of heat necessary to melt a gram of ice at zero, and m' the mass of ice melted, then,

$$mst = Lm',$$

or,

$$s = \frac{Lm'}{mt}. \tag{5}$$

The value of L could be found by introducing a mass of boiling water into the chamber, or by the method of mixtures thus: It has been found that if 150 gms. of water at 100° be mixed with 100 gms. of ice at 0° the ice will be melted and that the temperature assumed by the mixture will be about 28.3° C. As the hot water lost 10,755 cal., and the water of the melted ice gained 2830 cal., the difference, 7925 cal., must have been required to melt 100 gms. of ice. Hence,

$$L = 79.25 \text{ cal. per gram.}$$

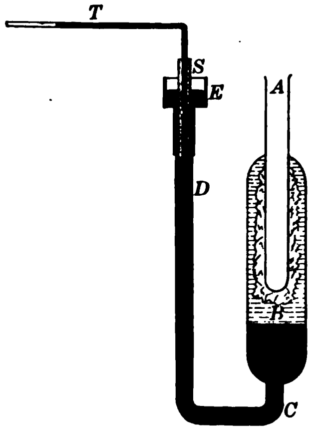


Fig. 146.

150. Bunsen's Ice Calorimeter. — A valuable form of ice calorimeter has been devised by Bunsen, in which the amount of ice melted is determined from the accompanying change of volume. The apparatus consists of a bent glass tube, CD (Fig. 146), furnished with a large bulb, B, into which is

sealed a test-tube, *A*. The other end of the stem is terminated by an iron collar, *E*, fitted with a stopper, *S*, in which is inserted a graduated tube, *T*, of fine bore. The collar *E*, the tube *CD*, and a portion of the bulb *B* are filled with mercury. The remainder of the bulb contains water carefully freed from air.

To prepare the apparatus for experiment a current of alcohol at a temperature below zero is passed through the test-tube *A* until the greater part of the water in *B* is frozen. Since water in freezing expands nearly a tenth, some of the mercury will be driven out into the stem *S* during the process. By adjusting the stopper in the mouth of the tube *D*, the extremity of the mercury column may be brought to any convenient point in the graduated stem. Some pure water is now poured into the test-tube and the apparatus packed in freshly fallen snow, till the temperature becomes zero throughout.

In order to determine the thermal equivalent of a scale division, a gram of water at 100° C. is dropped into the tube. As it gives up its heat, some of the ice will be melted and the end of the mercury thread will retract. Suppose that the reading on the scale before the hot water was put in is *a*, and that after the temperature has again fallen to zero it is *b*; a scale division will, then, correspond to

$$\frac{100}{b-a} \text{ calories.}$$

If, now, a gram of the substance to be tested, at a temperature of 100° C., be dropped into the tube, more ice will be melted, and the mercury thread further retracted to some point, *c*. The total heat abstracted from the body will be

$$\frac{c-b}{b-a} 100 \text{ calories,}$$

and the specific heat

$$s = \frac{c-b}{b-a}.$$

This instrument is particularly valuable for measuring the specific heats of substances which are obtainable only in small quantities.

151. Steam Calorimeter. — A form of calorimeter depending on the condensation of steam has been recently introduced by Dr. Joly, and is found to give more accurate results than any method yet employed.

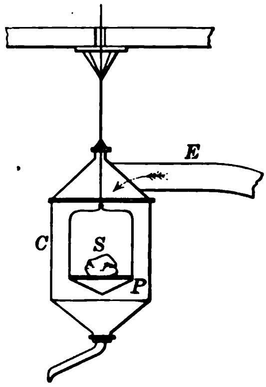


FIG. 147.

The apparatus consists, essentially, of a pan, P (Fig. 147), suspended from the arm of a balance, and surrounded with a chamber, C, to which steam may be admitted and allowed to escape at pleasure. In making an experiment a known mass of any substance is placed in the pan P, balanced, and allowed to attain the temperature of the chamber, which is read by means of a thermometer passing through its side. Steam is then suddenly admitted through the pipe E, so as to fill the chamber at once. As it condenses on the substance the water is collected in the pan P, and weights are added to the opposite pan of the balance to restore the equilibrium. A slow circulation of steam, not enough to disturb the equilibrium of the balance, is kept up till the substance has attained the temperature of the steam, when the condensation, of course, ceases.

Call T the temperature of the steam, L the latent heat of vaporization of water, and t the temperature of the chamber

when the steam was first admitted. Also let m be the mass of the substance, s its specific heat, M the amount of steam condensed, and k the thermal capacity of the pan P. Then the quantity of heat abstracted from the steam during condensation was ML, also the heat added to the substance was $ms\ (T-t)$, and the heat gained by the pan $k\ (T-t)$, whence

$$ms\ (T-t) + k\ (T-t) = ML. \tag{6}$$

The value of k is readily found from a separate observation.

To secure the greatest possible accuracy a small correction must be applied for the buoyancy of steam and of air.

The great merit of the steam calorimeter is that it may be used for bodies of any size and in any state, provided they are enclosed by a proper envelope of known thermal capacity.

152. Conditions which Affect the Specific Heat. — The specific heat of all bodies is more or less dependent on the temperature at which the body is examined. The law of variation is not known, but the results of experiment, in the vast majority of cases, may be expressed by a formula of the form

$$s = a + bt + ct^2 + \text{etc.},$$

where a, b, c, etc., are constants.

The increase of specific heat with rising temperature, in most solids, is small until the melting point is approached. Carbon is, however, a notable exception. Changes in density of solids produced by hammering, and the passage from one state to another, also largely influence the value of the specific heat. The allotropic varieties of calcium carbonate and carbon differ considerably in their specific heats.

SPECIFIC HEATS OF SOLIDS.

Aluminum	0.2122	Marble	0.216
Antimony	0.0507	Mercury	0.03192
Bismuth	0.0305	Nickel	0.1092
Brass	0.0939	Phosphorus	0.1699
Copper	0.0950	Platinum	0.0324
Glass	0.198	Quartz	0.19
Gold	0.0324	Silver	0.0599
Graphite	0.310	Steel	0.118
Ice	0.504	Sulphur	0.1844
Iron	0.1124	Tin	0.0559
Lead	0.0315	Zinc	0.0935
Magnesium	0.245		

SPECIFIC HEATS OF LIQUIDS.

Substance.	Specific Heat.	Temp. C.°
Alcohol	0.5475	0
Alcohol	0.6479	40
Ether	0.5290	0
Ether	0.5468	30
Mercury	0.0333	30
Turpentine	0.4537	40

153. Specific Heat of Water. — An exact knowledge of the specific heat of water is of fundamental importance in the study of calorimetry, because all investigators have used as their unit of heat the thermal capacity of some mass of water, though at various temperatures. The fact that the specific heat of water varied with the temperature was ascertained in the early part of the century, but the exact nature of the variations was first established by the work of Rowland in 1879.

The experiments of this investigator were of great accuracy, and showed that the specific heat diminished by about

one per cent from 0° to 29°, where its value was a minimum, and then increased again to the boiling point.

154. Specific Heats of Gases. — The changes of volume which a solid or a liquid undergoes in a moderate rise of temperature are so small that no account need be taken of the pressure to which the body is subjected. In the case of a gas under constant pressure the expansion with rise of temperature is considerable, and the thermal capacity is intimately connected with the work done by this expansion. There will, accordingly, be two distinct values of the specific heat for a gas according as the pressure or as the volume is kept constant.

155. Specific Heat under Constant Pressure. — The specific heat of gases under constant pressure has been carefully determined by Regnault. His method was to pass a definite quantity of the gas, at a known temperature, through a series of spiral tubes surrounded by cold water. The elevation of the temperature of the water was taken as a measure of the heat given up by the gas in falling through an observed range of temperature. The researches of Regnault showed that for a body rigorously obeying Boyle's Law the specific heat at constant pressure is independent of the pressure and of the temperature.

SPECIFIC HEATS AT CONSTANT PRESSURE.

Air	0.2374	Bromine	0.0555
Oxygen	0.2175	Chlorine	0.1241
Hydrogen	3.4090	Nitrogen	0.2438

156. Specific Heat under Constant Volume. — No satisfactory direct determination of the specific heat at constant volume was made until the invention of the steam calorim-

eter of Joly. The experiments of this investigator indicate that the specific heat, when the volume is kept constant, increases with the density in the case of air and of carbon dioxide, but diminishes in the case of hydrogen. The chief results as far as published are:

	Pressure in Atmos.	Density. Grams per cu. cm.	Specific Heat at Constant Volume.
Air	19.51	0.0205	0.1721
Hydrogen	—	—	2.402
Carbon Dioxide . .	7.20	0.011530	0.16841
Carbon Dioxide . .	12.20	0.019950	0.17504
Carbon Dioxide . .	16.87	0.028498	0.17141
Carbon Dioxide . .	20.90	0.036529	0.17305
Carbon Dioxide . .	21.66	0.037802	0.17386

157. Atomic Thermal Capacities. Law of Dulong and Petit. — As early as 1819 attention was called by Dulong and Petit to a remarkable relation between the specific heat of a simple substance and its atomic weight.

Consideration of such a table as that on page 196 led these investigators to announce the law which now bears their name, *viz.*, that the product of the specific heat by the atomic weight is the same for all elementary substances. The truth of this law was later investigated by Regnault, who found that it held approximately for substances which occur in the solid state. For 32 of these substances the mean product was 6.38, the extreme values being 6.76 and 5.7. Some variation in the value of this product might be expected from the dependence of specific heat on temperature and molecular conditions, explained in Art. 152.

EXAMPLES.

1. 45.1 gms. of copper at a temperature of 99.6° was immersed in 52.5 gms. of water at a temperature of 10°. The temperature of the water after immersion rose to 16.8°. Required the specific heat of the copper. *Ans.* 0.0956.

2. 72.3 gms. of a substance having a specific heat 0.874, at a temperature 95.6°, was mixed with 129 gms. of a liquid at a temperature 23.7°. The temperature of the mixture was found to be 56.3°. What was the specific heat of the liquid? *Ans.* 0.590.

3. 937 gms. of a substance having a specific heat 0.787, at a temperature 15.3°, was mixed with 596 gms. of a second substance having a specific heat 0.568, and at a temperature 135°. What should be the resulting temperature? *Ans.* 52.9°.

4. 16.1 gms. of sand at a temperature of 75° and 20 gms. of iron at 44.9° are thrown into 50 gms. of water at 3.9°. What should be the temperature of the mixture? *Ans.* 9.9°.

5. 96.8 gms. of ice at 0° were thrown into 156 gms. of water at 98.3°. What should be the temperature of the mixture? *Ans.* 30.3°.

6. 1.08 kilos of iron at 100° were placed in an ice calorimeter and melted 155 gms. What was the specific heat of the iron? *Ans.* 0.114.

7. What should be the result of mixing 51 gms. of snow at 0° with 230 gms. of water at 20°? *Ans.* Water at 2.0°.

8. What should be the result of mixing 6.2 gms. of snow at 0° with 7.1 gms. of water at 50°? *Ans.* 1.7 gms. of snow, and 11.6 gms. of water.

9. What should be the result of mixing 30.2 gms. of snow at −10° with 79.8 gms. of water at 40°, the specific heat of snow being taken as 0.504? *Ans.* Water at 5.9°.

10. What change of volume would be produced in a Bunsen ice calorimeter by the insertion of 2.5 gms. of a substance at 100°, having a specific heat 0.076? *Ans.* 0.0217 cc.

11. 15.1 gms. of mercury at 100° produce a contraction of 0.0567 cc. in a Bunsen ice calorimeter. What is the specific heat of mercury? *Ans.* 0.0329.

CHAPTER XI.

CHANGE OF STATE.

158. Laws of Fusion.—When heat is continuously applied to an amorphous body, *e.g.* glass, resins, etc., the change from the apparently solid to the liquid condition takes place in a continuous manner; that is to say, there is no temperature at which the substance may be said to begin to melt. In the case of most crystalline substances, however, when under a definite pressure, the change from the solid to the liquid state, or *vice versa*, appears to occur at a fixed temperature and to be discontinuous.

At the temperature of fusion the body is capable of co-existent phases; *i.e.* each state in the presence of the other is stable, but at other temperatures only one of the states is so.

If a definite amount of heat be applied to a body at its temperature of fusion, a definite quantity of the solid will be melted; and, conversely, if the same amount of heat be abstracted from the liquid at that temperature, the same mass will be solidified.

159. Fusion of Alloys.—Alloys of two or more metals frequently have a melting point lower than that of any of the components. Thus, an alloy made 5 parts tin and 1 part lead fuses at 194° C.; another, called Rose's Fusible Metal, composed of 4 parts bismuth, 1 part tin, and 1 part lead, melts at 94° C.

Other physical and chemical properties of such alloys indicate that the atoms have grouped themselves in new ways which are hardly distinguishable from the chemical molecule.

TABLE OF MELTING POINTS.

	C.		C.
Aluminum	850°	Mercury	−39°
Antimony	432	Nickel	1450
Bismuth	268	Platinum	1775
Brass	1015	Silver	954
Copper	1054	Steel, Cast	1375
Gold	1045	Sulphur	115
Iridium	1950	Tin	233
Iron, Cast	1200	Zinc	433
Iron, Wrought	2000	Glass	1100
Lead	326	Paraffine	54

TABLE OF LATENT HEATS OF FUSION.

UNIT, CALORIE PER GRAM.

Bismuth	12.64	Platinum	27.2
Ice	79.25	Silver	21.07
Iron, Cast, Gray	23.0	Sulphur	9.37
Iron, Cast, White	33.0	Tin	14.25
Lead	5.37	Zinc	28.13
Mercury	2.83		

160. Unstable Condition at Melting Point. — It is frequently possible, by slow and undisturbed cooling, to reduce a liquid below the temperature of freezing before solidification will begin. Thus, water freed from air and covered with a layer of oil may be cooled to −12° C. without freezing. Similarly, drops of water suspended in a fluid of their own density have been reduced to −20° C. without change of state. If, however, a fragment of ice be dropped into water thus over-cooled, or if the vessel be jarred, solidification will begin at once.

161. Change of Volume at the Melting Point. — Bodies which have a definite melting point exhibit a more or less abrupt change of volume on liquefaction. A majority of

substances expand on melting, but there are many exceptions, the most notable among these being water, which increases in volume from 1 to 1.0907; that is, more than one-twelfth, at the moment of congelation. This property plays an important part in the economy of nature. Thus, ice, being less dense than water, floats on the surface of ponds and rivers, where it serves as a protection to aquatic animal and vegetable life. If the density of ice were greater than that of water, it is evident that the ice would sink as fast as formed, and the entire body of water would soon be frozen solid. The stress exerted by water in solidification is very great, and usually bursts any vessel in which it is allowed to freeze. The disintegration of rocks thus effected by the frost is an important step in the preparation of the soil.

Among the metals, iron, bismuth, and antimony either expand on solidifying or change very little, and are consequently well suited for castings, as they retain an exact impression of the mold.

TABLE SHOWING THE EXPANSION ON MELTING.

Metal.	Density of Solid.	Density of Liquid.	Percentage Change in Volume from Solid to Liquid.	
Bismuth	9.82	10.055	Decrease	2.3 %
Copper	8.8	8.217	Increase	7.1
Lead	11.4	10.37	"	9.93
Silver	10.57	9.51	"	11.2
Tin	7.5	7.025	"	6.76
Zinc	7.2	6.48	"	11.1
Cast Iron	6.95	6.88	"	1.02

162. Volume-Temperature Diagram of Water. — A valuable method of representing the various states that a body may pass through is to plot certain of the physical coördinates of the state of the body, such as volume and pressure,

or temperature, as geometrical coördinates on a diagram. Thus, let the abscissas in Fig. 148 represent temperatures, and the ordinates volumes of the unit mass of water under a constant pressure of one atmosphere. Choose the ordinate of the point C proportional to the volume of a gram of freezing water which is approximately 1.000116 cc. and let its abscissa be 0°. Then the point B, directly above C, and having an ordinate 1.0907 times as great, will represent the ice at zero. The contraction of the ice, as the temperature falls, is shown by the line BA, whose slope is given by

$$\tan\phi = 0.000457,$$

the coefficient of expansion for ice.

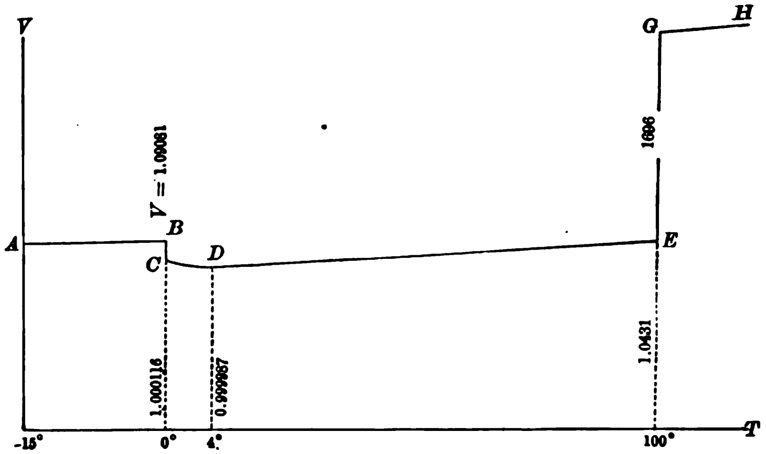


Fig. 148.

From C to D the water contracts, and from D to E it expands, according to the law given on p. 178. At E vaporization commences, and continues to G, the volume increasing 1696 times, but the temperature remains constant. The further expansion is along GH, having a slope

$$\tan\phi = \frac{1}{273} \text{ nearly.}$$

The condition of water which has been over-cooled would be represented by the curve *Ca*, Fig. 149.

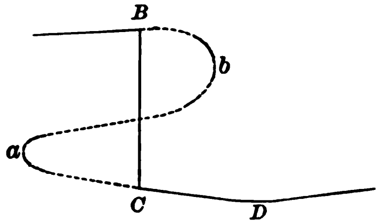


FIG. 149.

It is conceivable that the change from water *C* to ice *B* might take place along the path *CabB*, but since the states represented by the curve *ab* are in the region of complete instability there seems little probability that they can be observed.

163. Influence of Pressure on the Melting Point. — It was suggested in 1849 by James Thomson, from purely theoretical considerations, that water which contracts on liquefaction should have its melting point lowered by increase of pressure, by about 0.0075° C. per atmosphere. These predictions were shortly afterward verified by his brother, Lord Kelvin.

Without presenting the mathematical theory at this point, it is not difficult to see that if a body contract on liquefaction an increase of pressure would assist the change, and probably allow it to take place at a lower temperature. Likewise, if a body expand on melting, a large pressure should retard the change or raise the melting point. Bunsen found that paraffine which melted at 46.3° C. under atmospheric pressure must be raised to 49.9° C. before it began to fuse, if the pressure was 100 atmospheres. Experiments on spermaceti and stearin showed analogous effects.

164. Regelation. — When two pieces of melting ice are pressed together, they freeze along the surface of contact. This phenomenon, called *regelation*, may be explained by the principles of the preceding section. The pressure at the points of contact causes some of the ice to melt; the

water so formed escapes to a region where the pressure is less, and, since it is below zero, freezes again, cementing the blocks together. The following experiment, due to Bottomley, well illustrates the melting of ice under pressure and regelation. A block of ice, A (Fig. 150), is supported on two wooden bars, B, C, and a copper wire, with two weights, D, E, suspended from the ends, passed over the top. The wire will be found to work its way slowly through the block, and finally fall to the floor, but the section through which it has passed will be even more firmly frozen than at first. The pressure of the wire melts the ice immediately below it, and the water thus formed escapes to the upper side, where it at once freezes, as it is free from the pressure and below zero. Incidentally, the experiment illustrates the principle demonstrated in Art. 107, that the pressure beneath the wire varies directly as the curvature. For whatever the form of the upper surface of the block may be when the experiment is started, the wire ultimately assumes a circular arc within the block, thus showing that, if the curvature is at first greater at one place than another, the more rapid descent of the wire at that point has the effect of diminishing the curvature until all parts progress uniformly.

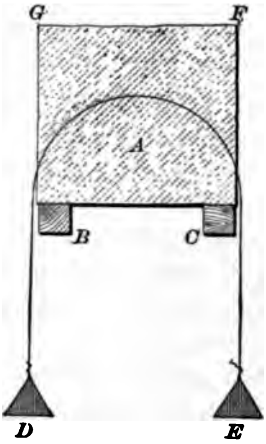


FIG. 150.

The formation of a snowball by squeezing moist snow between the hands is another example of the melting of ice under pressure and regelation. If the snow is too cold, the ball refuses to "make," for the reason that the pressure of the hand is not sufficient to bring it to the melting point.

In like manner small pieces of ice placed in a mold may be squeezed into a nearly homogeneous mass of ice, having exactly the shape of the mold, if they are subjected to sufficient pressure.

An example of the same phenomenon, and of great interest on account of its intimate connection with the climate of every part of the world, is afforded by the formation and flow of glaciers. Large quantities of water evaporating in the warm regions are carried toward the poles by the prevailing winds of the upper air and deposited as snow within the arctic circles. As the temperature rarely rises to the melting point, the snow collects in immense amounts. The effect of the weight of the upper layers is to compact the lower portion into a solid mass of ice, from which small streams of water flow out.

If, now, the ground slope toward the sea, as is generally the case in the valleys where the greatest quantities of snow collect, the relief of the pressure by melting at one point may subject another part to stress sufficient to break it. Thus, as a result of the breaking, melting, and freezing, the glacier creeps slowly toward the sea. If it were not for this method of return of the snow to the ocean, it seems probable that the accumulation of vast quantities in the polar regions would affect the stability of rotation of the earth. The glaciers which occur in high altitudes of the temperate zone have been studied by many investigators. Those of Switzerland are found to have a velocity of one or two feet per day.

165. Evaporation. — According to the molecular theory of the constitution of bodies presented in Art. 72, the molecules of a liquid, while not restricted to a definite position, being free to wander throughout the body, are, nevertheless, within the sphere of action of other molecules, for bodies in

this state resist forces acting so as to increase their volume. If, however, a molecule happen to be moving with sufficient velocity near the free surface of a liquid, it may escape entirely from the region of attraction of the other molecules, and move in a straight path till it meets some obstacle. It is now said to be in the aeriform condition, and this change of state is called *evaporation.*

166. Sublimation. — When a body passes by evaporation from the solid to the aeriform condition, the mode of change is termed *sublimation.*

At atmospheric pressure, camphor, arsenic, and many less familiar substances volatilize without passing into the liquid condition. If, however, the pressure be sufficiently increased, they may be fused. A light fall of snow is often seen to disappear by sublimation, when the temperature is considerably below freezing. Experiment shows that for a number of substances there is a definite pressure peculiar to each, below which it is impossible to melt the body. It has been proposed to designate this pressure as the critical pressure. This pressure must not, however, be confused with the pressure of the critical state (Art. 180).

167. Ebullition. — If heat be applied to a liquid under constant pressure in an open vessel, the temperature will rise, the average velocity of the molecules will be increased, and in consequence evaporation will go on at a more rapid rate. When a certain temperature, depending on the substance and the pressure, has been reached, it ceases to change, bubbles begin to form at the sides of the containing vessel and rise through the liquid, growing rapidly in volume. The substance is then said to be in *ebullition*, and the constant temperature at which the vaporization goes on is called the boiling point.

TABLE OF BOILING POINTS.

UNDER A PRESSURE OF 760 MM. OF MERCURY.

SUBSTANCE.	BOILING POINT C.°	SUBSTANCE.	BOILING POINT C.°
Alcohol . . .	77.9	Nitrogen . . .	– 194
Ammonia . . .	– 38.5	Oxygen . . .	– 184
Carbon Dioxide .	– 78.2	Sulphur . . .	448.4
Chlorine . . .	– 33.6	Sulphuric Acid .	338
Hydrogen . . .	– 243	Zinc	950
Ether	34.9	Sulphur Dioxide	– 10.5
Mercury . . .	350	Oil of Turpentine	159.3

168. Latent Heat of Vaporization. — By a specially contrived apparatus, too complicated for complete description here, Regnault found that the quantity of heat necessary to change a gram of water into steam at the same temperature could be expressed with considerable accuracy by the formula,

$$L = 606.5 - 0.695\,t,$$

where t is the temperature at which the change takes place. When the variation of the specific heat of water is taken into account, a small correction must be applied. Thus, the latent heat at 100° C. is more exactly 536.5 calories per gram, and at 200° C. 464.3 calories per gram.

TABLE OF LATENT HEATS OF VAPORIZATION.

SUBSTANCE.	CALORIES PER GRAM.	TEMPERATURE C.°
Alcohol	202.4	77.9°
Bisulphide of Carbon . . .	86.7	46.2
Ether	90.4	34.9
Mercury	62.0	350
Oil of Turpentine	74.0	159.3
Water	536.5 Regnault 535.9 Andrews	100

TABLE SHOWING CHANGE OF VOLUME ON VAPORIZATION.

AT A PRESSURE OF ONE ATMOSPHERE.

SUBSTANCE.	$\frac{\text{VOL. VAPOR.}}{\text{VOL. LIQUID.}}$
Alcohol	528
Ether	298
Oil of Turpentine	193
Water	1696

169. Unstable Condition at the Boiling Point. — As early as 1777 attention was called to a variation of the boiling point by the discrepancies in the reading in thermometers on which the upper fixed point had been determined by immersion in water boiling under the pressure of one atmosphere. The explanation of these variations is to be found in the existence of an unstable condition in the region of the boiling point, quite analogous to that presented by substances near the temperature of liquefaction. The presence of nuclei about which vapor may begin to accumulate, and more especially the presence of dissolved air, which itself collects in small bubbles when the temperature rises, greatly favors the beginning of ebullition, so that in proportion as the liquid is freed from these, the boiling point is raised. When retarded ebullition does commence it goes on in an almost explosive manner, and the liquid bumps violently against the bottom of the vessel. Dufour, by suspending drops of water in a mixture of linseed and clove oils, found that a drop 10 mm. in diameter could be raised to a temperature of 120° C., while drops 1 mm. in diameter remained liquid up to 178° C. When touched with a glass rod, or by the side of the vessel, they exploded at once.

170. Spheroidal Condition. — If drops of water be placed on a metal plate which has been heated to a proper temper-

ature, they do not vaporize at once, but roll about in little globules after the manner of mercury, or gather together in one mass which vibrates through an elliptical or stellate form. The phenomenon puzzled the first observers, who thought that the liquid had assumed a new form which they named the *spheroidal state.* Careful examination showed that there was no contact between the plate and the drop, thus suggesting the true explanation, namely, that the drop is supported on a cushion of its own vapor, which, being a poor conductor of heat, permits only a slow evaporation. The form assumed by the drop is fully accounted for by the laws of surface tension.

171. Cooling by Evaporation. — If it be assumed that heat is the kinetic energy due to the irregular motion of the molecules, it is easy to see that, whenever a molecule near the surface of a liquid escapes into a free space above, the kinetic energy of the liquid must be diminished by the amount of work done in separating the molecule from the liquid and by the energy which the molecule takes with it; that is to say, the liquid will be cooled by evaporation.

When the process is sufficiently rapid and long continued, the liquid may be frozen. This was first performed by Leslie, who supported a small copper dish containing water over a vessel of sulphuric acid, and placed the whole under the receiver of an air pump. On exhausting the air the water evaporates rapidly, but the vapor is continuously absorbed by the sulphuric acid. The temperature consequently falls, and, if the process be conducted with sufficient celerity, it is quite possible to have the water boiling and freezing at the same moment.

Very low temperatures may be produced by the vaporization of the more volatile substances. Thus, by opening a

small orifice in a vessel containing liquid CO_2, the escaping spray cools by evaporation so rapidly that it freezes in the form of fine snow.

172. Triple Point. — Let a curve be drawn on the volume temperature diagram representing the changes which a body undergoes while the pressure remains constant at p_1, for such a substance as water.

Thus, in Fig. 151,

a_1b_1 represents the expansion of the solid,
b_1c_1 " " contraction during melting,
c_1d_1 " " changes in volume in the liquid,
d_1e_1 " " expansion during vaporization,
e_1f_1 " " " in the aeriform condition.

For a pressure p_2 less than p_1 the analogous changes are represented by the line $a_2b_2c_2d_2e_2f_2$, in which b_2c_2 and e_2d_2 have approached each other. If the pressure is lowered sufficiently, these lines will ultimately meet in some line, such as BC; that is to say, by lowering the pressure the freezing point is raised and the boiling point lowered, until a condition is reached in which water in the solid, liquid, and aeriform states may exist together in equilibrium. This state, realized in the experiment of Leslie (Art. 171), is known as the *triple point*.

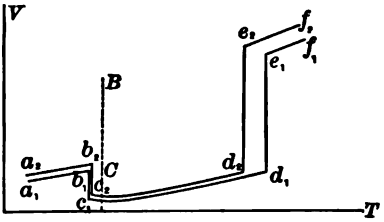


FIG. 151.

173. Cryophorus. — An instrument for showing the freezing of water by evaporation, invented by Wollaston and called the *cryophorus*, is figured in the accompanying sketch, Fig. 152. It consists of two glass bulbs, A, B, connected by a bent

tube. A small quantity of water is introduced, and, after boiling to expel the air, the whole is hermetically sealed. The experiment is commenced by running all the water into one bulb and immersing the other in a mixture of snow and salt, which has a temperature of $-18°$ C. The condensation of the vapor in *A* lowers the pressure in *B*, permitting a rapid evaporation from the water, which in time freezes solid. The bulb is not fractured by the expansion of the ice, as might be expected, because the ice itself volatilizes so rapidly that complete contact at the sides of the bulb never occurs.

174. Freezing Machines. — The principles just explained have been applied to the manufacture of ice and to refriger-

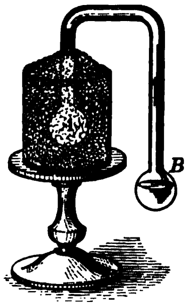


FIG. 152.

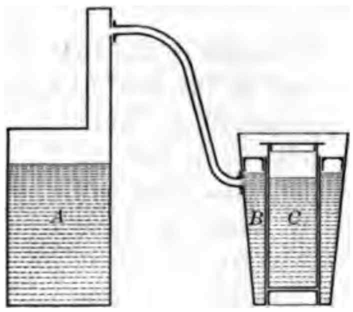


FIG. 153.

ation on a large scale. Some of the best-known systems make use of ammonia, according to a method introduced by Carré in 1860. The apparatus consists essentially of a boiler, *A* (Fig. 153), nearly filled with a strong solution of aqua ammonia, and connected by a pipe with the chamber *B*, which surrounds the vessel *C* containing the water to be frozen. When heat is applied to the boiler, the ammonia gas is driven over into *B*, where it is condensed in the presence of a small quantity of water. Connection with *A* is

now closed, and communication opened to a condenser, permitting a rapid evaporation of the ammonia gas from *B*, with consequent fall of temperature and freezing of the water in *C*.

175. Saturated Vapor. — When a liquid is placed in a closed, empty space, evaporation at first will go on rapidly; but as the space above becomes filled with molecules rebounding from one another and from the sides of the vessel, some of them are reflected into the liquid and remain there. In time the vapor and its liquid come into a state of equilibrium, *i.e.* as many molecules are returned through the free surface as escape from it in a given time. The vapor is then said to be *saturated.*

Fig. 154.

176. Pressure of a Saturated Vapor. — The pressure which any vapor is able to support cannot exceed a certain maximum amount, depending on the temperature and the nature of the substance. Any attempt to compress the vapor beyond this point (saturation) will result in the condensation of a portion into the liquid state. This behavior of a vapor may be easily shown by means of a barometer tube, *A* (Fig. 154), immersed in a cistern of mercury, *B*. If a few drops of ether be introduced at the bottom of the tube, it will rise to the top and quickly evaporate, producing a fall of the mercury below the true barometric hight *d*, proportional to the pressure exerted by the

vapor in the space *bc*. If the tube be now pushed further down into the cistern, the volume of the vapor will decrease, and its pressure, which is measured by the fall of the level of the mercury *c* below *d*, will increase until liquid ether begins to collect at the top of the mercury. On lowering the tube further, or raising it again, the level of the column will remain invariable as long as any liquid is present, the only observable effect being an increase or diminution of the amount of liquid ether. The condition of a saturated vapor is thus seen to depend on but two coördinates, the temperature and the pressure, instead of on three which are usually required to determine the state of a body.

TABLE OF MAXIMUM PRESSURES OF VAPORS.

IN DYNES PER SQ. CM.

TEMP. C.°	ALCOHOL.	ETHER.	CARBON DISULPHIDE.	CHLOROFORM.
– 20	4455	9.19×10^4	6.31×10^4	
– 10	8630	1.53×10^5	1.058×10^5	
0	16940	2.46 × "	1.706 × "	
10	32320	3.826 × "	2.648 × "	
20	59310	5.772 × "	3.975 × "	2.141×10^5
30	1.048×10^5	8.468 × "	5.799 × "	3.301 × "
40	1.783 × "	1.210×10^6	8.240 × "	4.927 × "
50	2.932 × "	1.687 × "	1.144×10^6	7.14 × "
60	4.671 × "	2.301 × "	1.554 × "	1.007×10^6
80	1.084×10^6	4.031 × "	2.711 × "	1.878 × "
100	2.265 × "	6.608 × "	4.435 × "	3.24 × "
120	4.31 × "	1.029×10^7	6.87 × "	5.24 × "

177. Vapor Pressure of Water. — The pressures of saturated water vapor have been studied by Regnault, who made use of two barometer tubes surrounded by a bath whose temperature could be varied. One of the tubes (Fig. 155)

contained only mercury; the other had a small quantity of water at the top of the mercury column. The difference of these columns, when corrected for capillarity and the expansion of the mercury, gave the value of the pressure sought. The results obtained are exhibited by the table on the following page.

FIG. 155.

It will be observed that the vapor pressure at the boiling point, 100° C., is just one atmosphere, as it should be; for the boiling point of any liquid is obviously that temperature at which the vapor in a bubble is able to maintain itself against the external pressure on the liquid.

178. Isothermals of a Gas. — If the pressure and volume be plotted as coördinates on a diagram, the line which represents the succession of states through which a body may pass while its temperature remains constant is called an *isothermal.* In the case of a gas the isothermals will be a series of equilateral hyperbolas, as appears from the equation

$$pv = mR\tau = \text{constant for } \tau \text{ constant.}$$

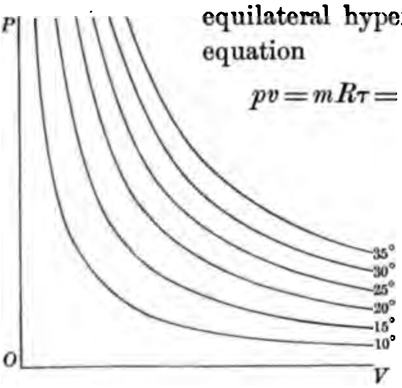


FIG. 156.

Fig. 156 shows a series of such isothermals drawn for intervals of five degrees. It is evident that the isothermal for any temperature must lie entirely above that for a lower temperature, since both the pressure and the volume increase with increasing temperature.

TABLE OF PRESSURES OF SATURATED WATER VAPOR.

In Centimeters of Mercury.

$$\left(1 \text{ cm. merc.} = 1.334(10)^4 \frac{\text{dynes}}{\text{cm.}^2}.\right)$$

Temp.	Pressure.	Temp.	Pressure.
− 30° C.	0.0386	100° C.	76.00
− 25	0.0605	102	81.60
− 20	0.0927	104	87.54
− 15	0.1400	106	93.83
− 10	0.2093	108	100.5
− 5	0.3113	110	107.5
0	0.4600	112	115.0
5	0.6534	114	122.8
10	0.9165	116	131.1
15	1.269	118	139.9
20	1.739	120	149.1
25	2.355	122	158.8
30	3.154	124	169.1
35	4.183	126	179.8
40	5.491	128	191.1
45	7.139	130	203.0
50	9.198	135	235.4
55	11.75	140	271.8
60	14.88	145	312.6
65	18.69	150	358.1
70	23.31	155	408.9
75	28.85	160	465.2
80	35.46	165	527.5
85	43.30	170	596.2
90	52.54	175	671.7
92	56.67	180	754.6
94	61.07	185	845.3
96	65.75	190	944.3
98	70.72	195	1052
99	73.33	200	1169

179. Isothermal of a Vapor. — Suppose that a given mass of aqueous vapor be placed in a cylinder furnished with a piston, and be kept at the constant temperature of 100°.

Let the pressure be represented by Oa'', and the volume by Oa', Fig. 157. If the vapor be now compressed, the temperature remaining constant at 100°, the pressure will increase, but not so rapidly as for a gas, since pv is something less than the constant value $mR\tau$. This series of changes will be represented by the curve ab, having an equation

$$(7) \qquad pv = B\tau - Cp^{\frac{1}{4}},$$

where B and C are constants. At the point b the vapor has reached saturation, and further compression is followed by condensation, the pressure all the while remaining the same until the point c is reached, where all the mass is liquefied. At any point between c and b, as at e, the proportion of liquid to vapor is given by $\frac{eb}{ce}$. As the liquid is very incompressible, from the point c an immense increase of pressure corresponds to but a very small diminution of the volume.

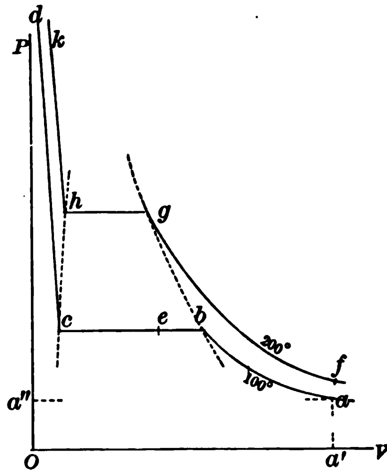


Fig. 157.

If this process of compression were to be repeated at a higher temperature, say 200° C., a new isothermal, $fghk$, would be obtained. The point of saturation g now has a greater pressure and a smaller volume. The point of complete liquefaction h, on the contrary, has a larger volume

than c, for the liquid expands with rise of temperature. The locus of the points of saturation for intermediate temperatures, sometimes called the *steam line*, is dotted from b to g. Another line, hc, passing through the points of complete liquefaction, may be called the *water line*. It is then evident that any point to the left of the water line refers to water in the liquid state; a point between hc and gb refers to mixed liquid and vapor, or the condition of coexistent phases; finally, a point to the right of the steam line refers tô the aeriform condition only.

180. Continuity of the Liquid and Aeriform States. — The first experiments which penetrated into the region lying above gh, Fig. 157, were made by Cagniard de la Tour in 1822, who found that when a liquid was enclosed in a glass tube, with less than half its volume of vapor, and raised to a certain temperature, the surface of demarcation between the liquid and vapor gradually lost its curvature and then disappeared entirely. Upon lowering the temperature the vapor in the tube assumed a cloudy appearance with flickering striae, after which the substance returned to its original condition of coexistent liquid and vapor. In 1823 Faraday succeeded in liquefying at ordinary temperatures, by pressure alone, chlorine and other substances previously known only in the gaseous state, and reached the important conclusion that no amount of pressure would produce the phenomenon of condensation at a temperature above a certain point peculiar to each body. The significance of these experiments of Faraday and Cagniard de la Tour was not well understood until the careful measurements on carbon dioxide by Andrews, in 1869, cleared the whole subject. The results of Andrews' experiments are plotted on the pressure-volume diagram of Fig. 158. At a temperature of 13.1° C. conden-

sation began when the pressure reached 49 atmospheres. The corresponding isothermal for air is drawn in the upper part of the figure.

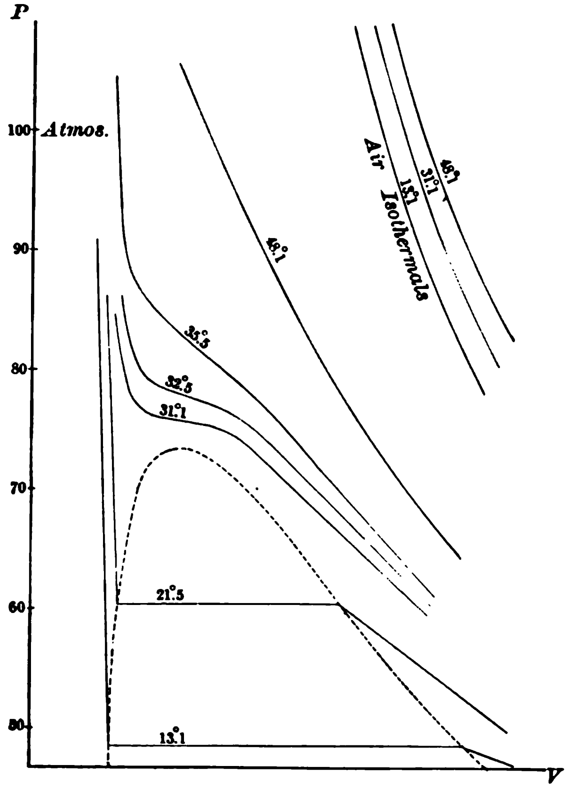


FIG. 158.

As the volume is diminished, the substance is gradually changed into the liquid condition. In this state the compressibility is small, but rather greater than for ordinary liquids. The experiment made at 21.5° showed the same general phenomena. Condensation did not begin till the

pressure reached 60 atmospheres. The volume of saturated vapor at this temperature is seen to be but little more than $\frac{3}{5}$ of that at 13.1° C., while the volume of the substance in the liquid state is considerably greater than at the lower temperature, thus indicating that liquid carbon dioxide has a large coefficient of expansion.

In the next experiment, at 31.1° C., Andrews found that the appearance of the vapor remained homogeneous throughout up to a pressure of 85 atmospheres. The isothermal, however, exhibits a marked inflection in the region of 75 atmospheres. Further examination showed that the vapor and liquid could not exist together above 30.9° C. which Andrews proposed to call the *critical temperature* for carbon dioxide. Above this temperature the ordinary distinctions of a liquid and a vapor have disappeared, while the substance partakes of the nature of both. The isothermals for 32.5° and 35.5° resemble those for 31.1°, except that the change in the compressible region is less abrupt. At 48.1° the inflection has entirely disappeared, and the form of the curve is approaching that of the air isothermal.

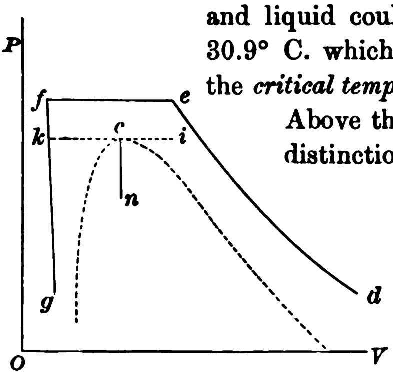


FIG. 159.

By a properly chosen cycle of changes, carbon dioxide may be made to pass from its ordinary or gaseous condition into the liquid state by an entirely continuous process. Thus, taking an amount of the gas in the condition denoted by *d* (Fig. 159), let it be compressed at constant temperature until it reaches *e*. If the temperature then be lowered, under constant pressure, the substance will pass to the state *f*, from which by isothermal expansion it may be brought to

the familiar liquid condition g, the entire passage from d to g having been accomplished by continuous changes.

The junction of the steam and liquid lines of a vapor is shown by the dotted curve in Fig. 158. A body in the condition represented by the highest part, c, of this curve (Fig. 159) is said to be in the critical state. By lowering the temperature under constant volume, the body passes into the region of mixed liquid and vapor. If the temperature be lowered under constant pressure, the state k approaches very closely the liquid condition; but if it be raised, under the same restriction, to i, the change is toward the gaseous state.

TABLE OF CRITICAL TEMPERATURES AND PRESSURES.

SUBSTANCE.	CRITICAL TEMP. C.°	CRITICAL PRESS.
Alcohol	234.3° 243.6	62.1 atmos. 62.8
Carbon Bisulphide .	271.8 273	74.7 77.9
Carbon Dioxide . .	30.92	73
Ether	190	36.9
Hydrochloric Acid .	51.25	86
Water	365	200.5

181. Continuity of State below the Critical Temperature. — It was suggested by James Thomson that the portions AB and EF of the isothermal (Fig. 160) should be joined by an ideal branch, BCD, making the change of state continuous, as above the critical temperature. A small

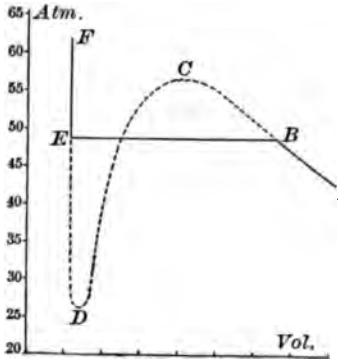


FIG. 160.

portion of DE can be realized, giving an unstable state, and the fluctuation of the boiling point mentioned in Art. 169. A phenomenon of supersaturation is also known, showing a real existence of the curve in the region of B. Experimental evidence of the portion DC can hardly be expected, since the pressure and the volume increase together, *i.e.* the state is one of complete instability.

182. Compressibility of Aeriform Bodies. — It was explained in Art. 69 that the distinction between a gas and a vapor was not absolute, but one of convenience. It is, therefore, necessary to inquire within what limits, and to what degree of accuracy, the bodies commonly known as gases obey Boyle's Law. Boyle's own experiments extended only from $\frac{1}{30}$th of an atmosphere to 4 atmospheres at ordinary temperatures. Thc first extended and critical examination of the compressibility of gases was made by Regnault, who showed that for the first 20 atmospheres air and nitrogen were more compressible than Boyle's Law would require, but that hydrogen was less so. Then Natterer, by a series of rough experiments with high pressures, sometimes as great as 3000 atmospheres, found that all of the three substances named were more compressible than Boyle's Law indicated, and that the deviation increased with the pressure.

The subject was allowed to rest until 1870, when Amagat, choosing a coal pit for his workroom, because its temperature would remain sensibly constant, and making use of a steel manometer tube filled with mercury and extending up the shaft a distance of 330 meters, secured a most interesting series of observations on hydrogen, nitrogen, carbon dioxide, and ethylene.

In order to compare the various substances, Amagat plotted the pressures as abscissas, and the product of the

pressure by the volume as ordinates. Fig. 161 shows his diagram for hydrogen. The lines for the different temper-

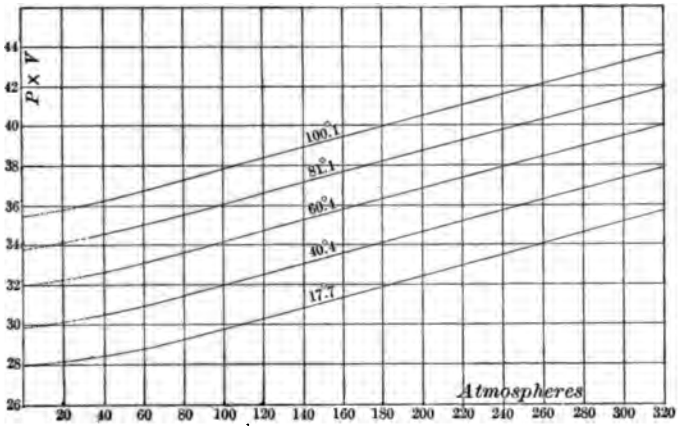


FIG. 161.

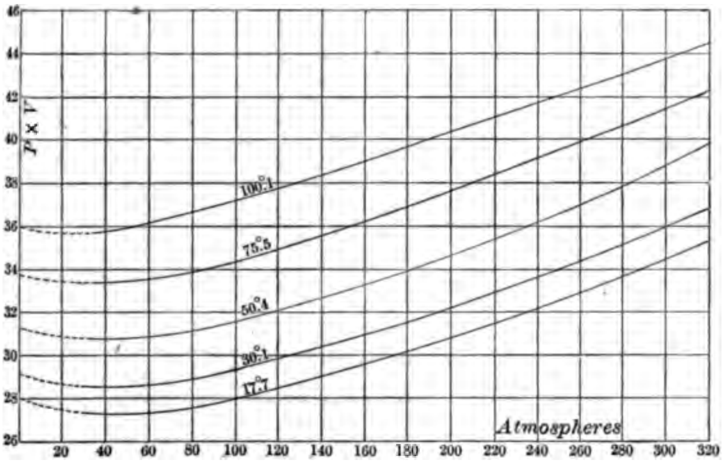


FIG. 162.

atures are parallel and nearly straight, but inclined to the axis of pressure in such a way as to show that pv increases

uniformly with the pressure. If Boyle's Law were fulfilled, the lines should be horizontal and straight.

The departure from a right line, which is just perceptible in the case of hydrogen, is more marked in the diagram of nitrogen (Fig. 162), where *pv* reaches a minimum in the vicinity of 40 atmospheres and afterwards rises.

The variations, which are but little more than suggested in the diagrams of hydrogen and nitrogen, are exhibited in their entirety by the diagram of carbon dioxide (Fig. 163).

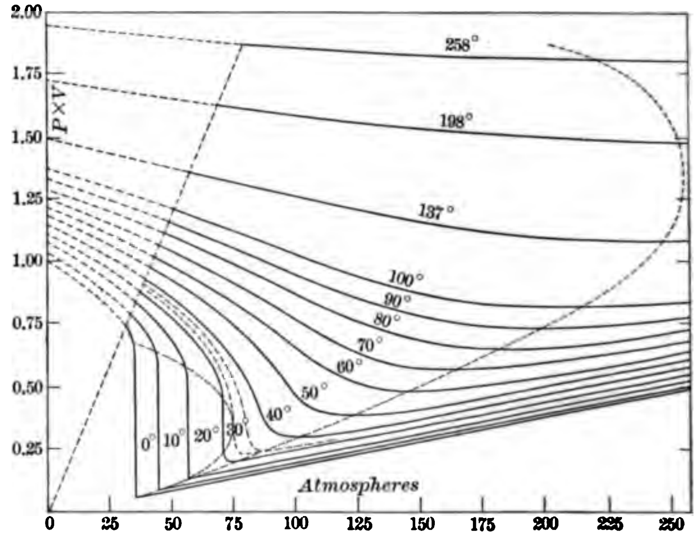


FIG. 163.

The points of minimum *pv* move toward the right, with rise of temperature, until about 180°, where they begin to move back.

Since condensation takes place below 30°, the values of *pv* are discontinuous in this region.

Little is known of the product of the volume and pressure of gases at low pressures, since the column of mercury,

by which the latter is measured, is too short to be read accurately, and the presence of the mercury vapor itself introduces considerable uncertainty. It appears probable, however, that the departure of this product from a constant value is very slight.

EXAMPLES.

1. Steam at 100° is passed into 120 gms. of water at 10° until the temperature rises to 35° and the mass of water has increased to 125 gms. What is the latent heat of steam? *Ans.* 535 cal. per gm.

2. 150 gms. of lead at 401° were dropped into 844 gms. of mercury at 0°. What was the resulting temperature, assuming that the specific heat of melted lead is 0.0402? *Ans.* 88.4°.

3. What is the least quantity of water at 10° needed to condense 100 gms. of alcohol at 78° into liquid at 15°, assuming the mean specific heat of alcohol to be 0.65? *Ans.* 4.87 kilos.

CHAPTER XII.

SOLUTIONS.

183. Definition of a Solution. — In the most general sense a solution may be defined as a homogeneous mixture which cannot be separated by mechanical means.

In the aeriform condition bodies mix in all proportions. In the liquid state the miscibility is limited in amount and dependent on the nature of the substances. The solution of one solid by another is not unknown, but the cases are of comparatively little importance. The substance which is present in a solution in the largest quantity is usually spoken of as dissolving the other and may be called the *solvent*. The substance which is dissolved will be termed the *solvend*.

184. Solution of Gases in Gases. — When a homogeneous mixture of two or more gases is formed, the properties of the mixture are found to be the sum of the properties of the components.

The law connecting the pressure of a mixture of gases with the pressures of the components was first established by Dalton, who found that the total pressure of a mixture was equal to the sum of the pressures which each of the gases would separately exert in the given space. Thus, for instance, suppose a number of gases having initially the volumes $v_1, v_2, v_3 \cdots$, and pressures $p_1, p_2, p_3 \cdots$, be placed in a vessel whose volume is V, then the pressure of the first gas expanded to this volume would be

$$\frac{v_1}{V} p_1,$$

by the law of Boyle. Similarly, the pressure of the second alone occupying this volume would be

$$\frac{v_2}{V}p_2,$$

and so on.

If, then, P denote the pressure of the mixture, by Dalton's Law,

$$(1) \qquad P=\frac{p_1v_1}{V}+\frac{p_2v_2}{V}+\text{etc.},$$

or,

$$(2) \qquad PV=p_1v_1+p_2v_2+\text{etc.}$$

The explanation of this law is, apparently, to be sought in the fact that the molecules are so distributed that they are able to influence each other only by their kinetic energies, and not at all by those properties which are characteristic of each substance. For a similar reason it might be expected that other properties of such mixtures would be the sum of the corresponding properties of the components. This prediction has been verified for refraction and absorption of light, and there is no reason to doubt that it holds true for all cases.

185. Solutions of Liquids in Aeriform Bodies. — The evaporation which takes place at the free surface of a liquid permits the formation of a solution of a liquid in all aeriform bodies. Experiments by Dalton on the evaporation of a liquid in contact with a gas showed that the vapor pressure of the liquid in a space occupied by a gas was the same as in a vacuum. Further investigations have shown that although the law is substantially accurate it cannot be regarded as more than a first approximation, or the limiting case. The deviation from the exact law is to be assigned,

1°, to the mutual actions of the molecules of the vapor and the gas, and 2°, to the lowering of the vapor pressure consequent upon the solution of some of the gas in the liquid. (See Art. 195.)

Of the solution of solids in gases nothing is definitely known, but it is probable that Dalton's Law is followed to the same extent as in the solution of liquids in gases.

186. Solution of Gases in Liquids. — Liquids dissolve all gases without exception. The solutions are conveniently divided into two classes, according as the gas may or may not be completely expelled by heating under diminished pressure.

The first class is subject to a law discovered by Henry, namely, that the mass of the gas dissolved by a given quantity of liquid is proportional to the pressure if the temperature remains constant.

Since the density of a gas varies directly as the pressure, Henry's Law may also be stated in this way: The volume of the gas which can be absorbed by a given quantity of liquid at a definite temperature is the same for all pressures.

The ratio of the volume of the gas absorbed to the volume of the absorbing liquid is called the *solubility;* thus, if

$V =$ volume of liquid,
$v =$ " " the gas,
$\lambda =$ solubility,

then,

$$\lambda = \frac{v}{V}. \qquad (3)$$

The value of the solubility is found to diminish with increasing temperature.

In solutions from which the gas cannot be driven off by heating, or lowering the pressure, certain chemical changes appear to have taken place. The discussion of such solutions belongs rather to Chemistry than to Physics.

187. Solutions of Liquids in Liquids. — The solution of one liquid by another is, in general, accompanied by a change of volume and of temperature. Thus, on mixing equal volumes of alcohol and carbon disulphide, the temperature

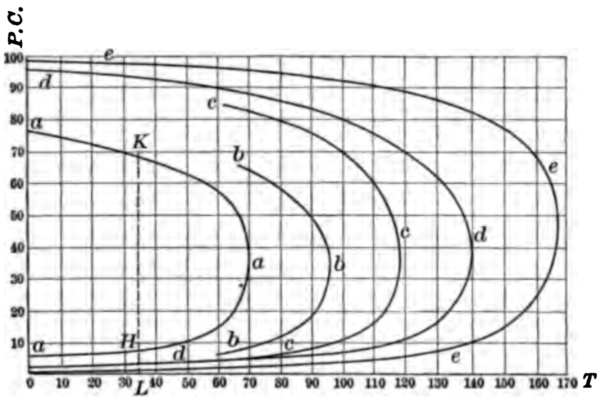


FIG. 164.

falls through 5.6° C. A mixture of ether and chloroform is accompanied by a rise of temperature of 14.4° C. and a diminution of volume. The amount and direction of these changes are frequently dependent on the proportion of the components in the mixture.

The proportion in which two liquids dissolve depends in a most important manner on the temperature.

Solutions of liquids in liquids are sometimes divided into classes, according as they are or are not miscible in all proportions. Alcohol and water, chloroform and carbon disulphide,

are pairs of liquids which are soluble in all proportions. On the other hand, certain pairs of liquids are soluble only within definite limits. For instance, if equal quantities of water and ether are shaken together, they will separate into two layers, the lower, an aqueous solution, containing 10% of ether, and the upper, or ethereal solution, containing 3% of water. If any liquid, *A*, is partially soluble in a liquid, *B*, then the reciprocal process will also obtain, and *B* will be partially dissolved by *A*.

A valuable method of recording experiments or solutions is to plot the temperatures as abscissas and the ordinates as the percentage of one substance in the solution. Thus, in Fig. 164 the curve *aaa* refers to water and phenol, *bbb* to water and salicylic acid, *ccc* to water and benzoic acid, *ddd* to water and aniline phenolate, *eee* to water and aniline.

It will be observed that at the low temperatures there are two definite proportions in which a stable solution is formed. For instance, *LH* is the proportion of phenol present when it is dissolved by the water, but *LK* is the amount of phenol in 100 parts of the solution when phenol is the solvent, the temperature in each case being 34° C. As the temperature rises, these proportions approach equality, and above this point the liquids are completely miscible.

Since there is no reason to doubt that all liquids which dissolve one another would show similar behavior at sufficiently high temperatures, the distinction made between two classes of such solutions is an entirely arbitrary one.

188. Solution of Solids in Liquids. — Many solids when brought in contact with a liquid are dissolved up to a certain limit, depending on the temperature, after which the process ceases. The liquid at this point is said to be saturated.

The concentration of any solution may be increased to saturation by lowering the temperature, after which some of the solvend will separate if a particle of the solid be present. In the absence of such a nucleus it is possible to cool the solution below the point of saturation without a separation of the dissolved solid. The solution is then said to be supersaturated.

The solubility of a solid in general increases with the temperature, but there are several important exceptions.

The volume of the solution formed by dissolving a solid in a liquid is usually less than the sum of the volumes of the components, by an amount depending upon the concentration.

189. Diffusion of Liquids. — When two different solutions are brought in contact, a slow change of concentration is observed to go on until the composition of the whole is homogeneous. This phenomenon, called diffusion, was first investigated by Graham in the following method. A glass jar, A, was filled with a salt solution and placed within a larger vessel, B. Water was then carefully poured into the latter until A was covered to a depth of about 3 cm. After standing for a considerable length of time, A was removed and the amount of salt present in the outer vessel determined. The chief conclusion deduced from such experiments was that the rate of diffusion varied greatly with the nature of the dissolved substance. Arranging the substances according to the rate of diffusion, Graham found that free acid bases and neutral salts were characterized by a much greater rate of diffusion than that shown by gums, tannin, albumen, and the like. As the substances of the first mentioned class are generally known in the crystalline form, he proposed to call them *crystalloids*. The second class are

amorphous and were named *colloids* (κόλλη = glue). Colloids are also distinguished by the fact that they are more or less impermeable by other colloids, but do not present any marked hindrance to the diffusion of crystalloids. Graham further utilized this principle to separate a mixture of crystalloidal and colloidal substances by a method which he called *dialysis*. The mixture is placed in a vessel and separated from pure water by a colloidal membrane, such as parchment or animal bladder. In course of time the crystalloids will diffuse into the water and leave the colloids behind.

190. Coefficient of Diffusion. — Let the concentration of a solution be defined as the mass of the solvend divided by the space through which it is uniformly distributed, and suppose that the concentration at one surface of a layer of liquid of thickness d is s_1, and the concentration at the other surface is s_2. Also suppose that a mass, m, of a substance diffuses through a cross section, A, of this layer in the time t, then the coefficient of diffusion K may be defined by the equation

$$m = KA\frac{(s_2 - s_1)}{d}t.$$

The dimensions of K are accordingly

$$[K] = \frac{L^2}{T}.$$

191. Diffusion of Gases. — Gases exhibit phenomena of diffusion identical with those just described in the case of liquids, but more marked in degree. If, for instance, a small quantity of a gas possessing a characteristic odor, *e.g.* chlorine, acetylene, etc., be liberated in one corner of a room, its

presence may be detected by the sense of smell in every part after a very short time.

When two gases are separated by a porous septum, the more rapid rate of diffusion of one of the gases through the partition may establish a temporary difference of pressure between the sides. This may be shown by means of a funnel, *A* (Fig. 165), closed at the top with a plate of plaster of Paris and connected at the bottom with a glass tube, *D*, dipping into a vessel of colored water, *C*. If a beaker, *B*, be inverted over the funnel and filled with hydrogen, or even ordinary illuminating gas, bubbles of air will begin at once to rise through the water in *C*. On removing the beaker, the liquid will rise in the tube to a certain hight, after which it will slowly fall to the original level. The diffusion of the rare gas into the funnel goes on so much more rapidly than that of the air outwards that the pressure rises and air is expelled from the tube. When the beaker is removed, the more rapid escape of the hydrogen causes the reverse effect in the tube *D*.

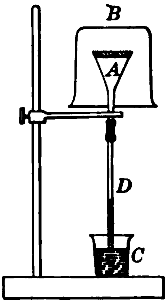


Fig. 165.

A mixture of gases may thus be separated by a method analogous to dialysis.

192. Osmose. — When two liquids are allowed to diffuse through a membrane which hinders the passage of one of the substances, a considerable difference of pressure may be established between the two sides. This mode of diffusion has received the name *osmose*, and the difference of pressure is called the *osmotic pressure*. It was first described by the Abbé Nollet, who found that if a glass vessel be covered with a bladder filled with spirits of wine and immersed in

water, the contents of the vessel would increase so as to expand the bladder, and sometimes even burst it.

Farther investigations have shown that this phenomenon has an important bearing upon the explanation of many processes in organic life which had been loosely attributed to a hypothetical vital force.

By the arrangement shown in Fig. 166, Dutrochet was able to obtain a rough measure of the osmotic pressure. *A* is a vessel closed at the bottom by a membrane, *C*, and fitted at the top with a long open tube, *B*. When this vessel *A* is filled with a solution of gum and water and immersed in a larger vessel, *D*, containing water only, it is found that after a time the level of the fluid in *B* rises to a hight considerably above the level of that in *D*, and at the same time traces of the gum are perceptible in the water of *D*.

FIG. 166.

Quantitative measurements of the phenomenon by means of Dutrochet's apparatus are unsatisfactory, because an animal membrane only hinders but does not stop the passage of the dissolved substance. A great advance in the direction of measurement was made by Traube, who discovered that a pellicle formed by precipitation when two solutions are brought together is permeable by water but not by certain other substances, including the reagents by whose action the pellicle was formed. By the use of such membranes deposited on the inner surface of a clay cylinder which was afterward filled with different solutions and immersed in water, Pfeffer deduced the following conclusions:

1°. That the osmotic pressure depends on the nature of the substance. With one per cent solutions the following pressures were observed.

	Cm. of Mercury.
Cane Sugar	47.1
Dextrin	16.6
Niter	178
Potassium Sulphate	193
Gum	7.2

2°. That the pressure is proportional to the concentration.

3°. That the pressure for a given concentration increases regularly with the temperature.

4°. That the pressure is dependent on the nature of the membrane.

This last conclusion is untrustworthy, since it may be shown to involve a contradiction of the law of conservation of energy. The experimental result may be accounted for by supposing either that the membrane yielded under the considerable pressures to which it was subjected, or that the membrane was not quite impervious to the solvend.

193. Investigation of de Vries. — Pringsheim discovered, in 1854, that the protoplasmic contents of an organic cell contract when the cell is brought in contact with a salt solution, but that the cell wall retains its form. The explanation of the phenomenon was later found to be in the fact that the protoplasm is contained in a membrane which is permeable by water, but is impervious to most dissolved substances. When, on the other hand, such a cell is placed in a solution having an osmotic pressure equal to or less than that of its own contents, the protoplasm will remain in contact with the cell wall. Thus, by observing the behavior of the contents of the cell, it will be possible to test whether any given solution has a greater or less osmotic pressure than the cell, and to compare different solutions.

By means of such a method de Vries verified the result previously obtained by Pfeffer, namely, that the pressure is

proportional to the concentration of the solution, always supposed dilute, and determined solutions which contain quantities of substances dissolved in the proportion of their molecular masses exhibit equal osmotic pressures.

By a similar method Donders and Hamburger showed that solutions which exerted equal pressures at a lower temperature would also exert equal pressures at a higher temperature, *i.e.* that the increase of pressure was independent of the nature of the dissolved substance. Van 't Hoff has since shown, by the principles of thermodynamics, that the pressure is proportional to the absolute temperature.

194. Analogy of Osmotic to Gaseous Pressure. — The laws of osmotic pressure, found by different investigators, may be summarized as follows:

1°. The pressure is proportional to the concentration, *i.e.* inversely proportional to the volume in which a given quantity of the substance is dissolved.

2°. The pressure is proportional to the absolute temperature.

3°. Quantities of the dissolved substances which are proportional to their molecular masses exert the same pressure at the same temperature.

The analogy of these laws to those of gaseous pressure is seen to be complete. The first corresponds to the law of Boyle, the second to the law of Charles, the third to the law of Avogadro. (See Art. 255.) This analogy was first pointed out and insisted on by Van 't Hoff, who showed that the formula for a gas,

$$pv = mRT,$$

was applicable without change in the value of R. (See Art. 138.)

The laws of osmotic pressure in dilute solutions may accordingly be reduced to the following simple statement: Dissolved substances exert the same pressure in the form of osmotic pressure that they would if gasified at the same temperature without change of volume.

195. Vapor Pressure of Solutions. — The effect of a dissolved substance on the vapor pressure of a liquid was first observed by the rise it produced in the boiling point. Since the vapor pressure of the pure liquid at the higher temperature would be greater than that of the atmosphere, it follows that the presence of a dissolved substance lowers the vapor pressure.

The law of this change may be concisely stated in a formula first proposed by Raoul,

$$\frac{p-p'}{p} = c\frac{n}{N+n}, \tag{4}$$

where p is the vapor pressure of the pure liquid, p' is the vapor pressure after the addition of the substance, c is a constant nearly unity for dilute solutions, n is the number of molecules of the dissolved substance, and N the number of molecules of the solvent.

The formula 4 stated in words is, that the lowering of the vapor pressure bears the same ratio to the whole pressure that the number of molecules dissolved does to the total number.

196. Application to the Determination of Molecular Masses. — By a simple transformation, equation 4 may be put in a form useful for the calculation of molecular masses.

Let N = number of molecules in solvent,
n = " " " " dissolved substance,
M = mass of a molecule in solvent,
μ = " " " " " dissolved substance,
M = mass of solvent,
m = " " dissolved substance,

then

$$(5)\qquad \begin{cases} N = \dfrac{M}{\mathrm{M}}, \\[2ex] n = \dfrac{m}{\mu}. \end{cases}$$

Now, accepting c as unity, equation 4 yields

$$(6)\qquad \frac{N}{n} = \frac{p'}{p - p'}.$$

Substituting the values of (5),

$$(7)\qquad \frac{M}{\mathrm{M}}\,\frac{\mu}{m} = \frac{p'}{p - p'},$$

whence, solving,

$$(8)\qquad \mu = \frac{\mathrm{M}}{M}\,\frac{p'm}{p - p'},$$

by which the molecular mass of the dissolved substance may be found when the lowering of the vapor pressure is known. By this means the knowledge of molecular masses has been much extended, as all methods previously used were applicable only to such substances as could be gasified without change.

The law of Raoul does not hold for aqueous solutions of salts, strong acids, and bases, as it is found that the lowering of the vapor pressure in these solutions is considerably greater than is required by equation 4, as if the solvend

had been split up into molecules smaller than usual, *i.e.* dissociated. This hypothesis is corroborated by a characteristic peculiarity of such solutions, namely, that of being readily decomposed by the passage of an electric current, and also by certain peculiarities in their chemical reactions.

197. Freezing Point of Solutions. — For reasons that are easily shown the diminution of the vapor pressure of a liquid by the presence of a substance in solution has the effect of lowering the freezing point. The freezing point is, by definition, that temperature at which the solid and its liquid are both in equilibrium with its vapor. Suppose, now, a small quantity of some salt be dissolved in the liquid, lowering its vapor pressure. The equilibrium between the solid and the vapor will be destroyed. The solid will begin to distil over into the solution at the expense of the usual amount of the latent heat of fusion, and the temperature falls to some point where the solid and the solution are again in equilibrium with the vapor.

A good example of this process is seen in the familiar freezing mixture consisting of snow and salt, in which the fall of temperature is about 22° C., before equilibrium is again established.

A theoretical calculation of the depression of the freezing point has been worked out by Van 't Hoff.

Let $T =$ the freezing point on the absolute scale,

$\Delta\tau =$ the lowering of the freezing point,

$n =$ number of molecules of the dissolved substance,

$N =$ " " " " " solvent,

$\mu =$ molecular mass of dissolved substance,

$L =$ latent heat of fusion per unit mass,

$R =$ the constant in $pv = mR\tau$,

[redacted]

then Van 't Hoff's formula may be written

$$\triangle T = \frac{RT^2}{L\mu} \frac{n}{N}.$$

The results obtained by this formula are in substantial agreement with observation, at least for dilute solutions, but it is not applicable to electrolytes in which the depression is greater than would be given by the molecular mass of the substances.

198. Humidity. — Humidity, or the state of the air as regards moisture, is defined as the ratio of the mass of aqueous vapor actually present to the quantity required for saturation. The ratio of the masses is obviously the same as that of the densities, and also that of the pressures, if it be assumed as a sufficient approximation, that aqueous vapor obeys Boyle's Law.

A great humidity is a source of considerable bodily discomfort either in summer or winter, but a very exact knowledge of this state of the air is of little importance. The humidity may be determined in three different ways, known respectively as the Chemical, the Dew Point, and the Wet and Dry Bulb methods.

199. The Chemical Hygrometer. — In the chemical method a known volume of air is passed through a series of tubes containing pumice moistened with sulphuric acid, which absorbs the water vapor. The mass of the water thus collected, divided by the amount necessary for saturation of the same volume at the observed temperature, gives the humidity.

200. Dew Point Hygrometers. — The temperature at which the aqueous vapor of the air begins to condense is called the *dew point*. It depends only on the amount of vapor actually present.

A dew point hygrometer is essentially a device for cooling a body till dew or hoar frost begins to collect, at the same time indicating the temperature. A simple form of dew point hygrometer invented by Dines is shown in Fig. 167. *R* is a reservoir filled with ice-water, which may be allowed to flow through the pipe *D* into a chamber, *E*, and escape at *A*. This chamber is covered with a piece of black glass, *B*, and contains a delicate thermometer, *T*. On opening the cock *C*, the passage of the cold water through *E* lowers the

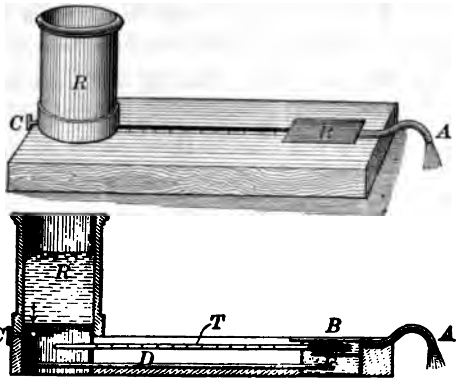


FIG. 167.

temperature of *B* until moisture begins to collect on its surface. The thermometer is then read and the flow of water stopped. The temperature of the glass soon begins to rise and the moisture to evaporate. Just as it disappears from the surface the thermometer is again read and the mean of both readings taken as the dew point.

It remains to show how the humidity may be calculated from the dew point. By the law of Dalton, given in Art. 184, the pressure of the aqueous vapor in the air may be regarded as the same as if the air were absent. Hence, the

only pressure which keeps the vapor next to the cold body *B* at the point of saturation is the pressure of the aqueous vapor in the uncooled air. And, *vice versa*, the pressure of saturated water vapor at the temperature of the cold body is that of the uncooled air. So that finally the humidity may be found by dividing the pressure of water vapor at the dew point by the pressure at the observed temperature of the air, their values being taken from the steam table on p. 216.

201. Wet and Dry Bulb Hygrometer. — The instrument most commonly used at meteorological stations for determining the humidity consists of two similar delicate thermometers fastened side by side on a stand (Fig. 168). The bulb of one of the thermometers, *W*, is covered with a piece of muslin which is kept constantly moistened by means of a cotton wick attached to the muslin and dipping into a small vessel filled with water. If the surrounding air is not saturated, evaporation will occur from the wet bulb, and the consequent abstraction of heat will lower the temperature below that of the surroundings. Thus, after a time, the wet thermometer will indicate a temperature constantly below that of the dry one by an amount depending on the humidity, since the rate of evaporation is determined by the amount of aqueous vapor present in the atmosphere.

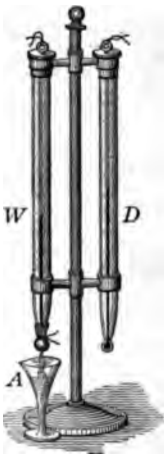


FIG. 168.

In order to deduce the dew point from the readings of the thermometers, recourse is had to a set of tables which have been constructed from a long series of simultaneous observations on the wet and dry bulb thermometers and a dew point hygrometer.

In dry weather, or when the air is quite still, the dew point, as deduced from observation of the wet and dry bulb thermometer, is usually a little high, probably because these conditions favor the elevation of the temperature of the wet thermometer by radiation from surrounding objects.

EXAMPLES.

1. 2.3 liters of hydrogen under a pressure of 78 cm. of mercury, and 5.4 liters of nitrogen at a pressure of 46 cm. were introduced into a vessel containing 3.8 liters of carbon dioxide under a pressure of 27 cm. What was the pressure of the mixture?

Ans. 140 cm. of merc.

2. The temperature of the dew point is observed to be 9.2° when the temperature of the air is 14.5°. What is the humidity?

Ans. 0.71.

3. When the humidity is 58% at the temperature 23°, what will be the dew point? *Ans.* 14.3°.

4. Calculate the weight of 15 liters of air at 20° saturated with water vapor, when the hight of the barometer is 75 cm. *Ans.* 17.66 gms.

CHAPTER XIII.

TRANSFERENCE OF HEAT.

202. Transference of Heat. — There are three ways by which heat may be transferred from one point to another.

1°. When heat is conveyed by the motion of portions of matter with which it is associated, the process is termed *convection.* It is illustrated in the familiar method of heating buildings by currents of hot air.

2°. When two parts of a body differ in temperature, and heat is transmitted from the warmer to the colder portions without perceptible motion of matter, the process is called *conduction.* If one end of a bar of iron be placed in a furnace and the other in a vessel containing ice properly screened from the furnace, the ice will be slowly melted by the heat which is conducted along the bar.

3°. When a hot body gives rise to a system of waves which traverse with a definite velocity space not filled with ponderable matter, and afterward warm a second body, the process of transfer is termed *radiation.* At no intermediate point, however, is the heat perceptible as such, but as a system of waves transmitted with the velocity of light, and afterward absorbed by the second body, giving rise to the observed thermal effects. The heat which reaches us from the sun is transmitted in this way.

203. Condition of Steady Flow in Conduction. — When a constant source of heat is applied to one part of a body, a general rise of temperature at other points will at first be noticed, but the temperature will finally become stationary

at values depending on the distance of the point considered from the source of heat. Under these circumstances there is said to be a steady flow of heat from regions of higher to those of lower temperature.

204. Definition of Thermal Conductivity. — Suppose two surfaces drawn in a body, on each of which the temperature has a constant but different value, say θ_1 on the first and θ_2 on the second. Then the quantity of heat Q which will pass in a time, T, through a small area, a, on each of these isothermal surfaces, where the distance between them is d, is found by experiment to be proportional to

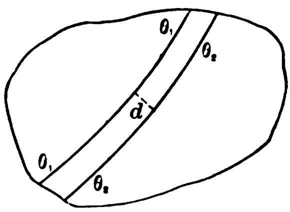


FIG. 169.

$$\frac{(\theta_2-\theta_1)\,aT}{d}$$

and dependent on the nature of the substance.

If Q be written

$$Q=k\,\frac{(\theta_2-\theta_1)\,aT}{d}, \tag{1}$$

the constant k is called the conductivity. Its value, as given by equation 1, is

$$k=\frac{Qd}{(\theta_2-\theta_1)\,aT}. \tag{2}$$

If heat is measured in calories, the dimensions of k are

$$[k]=\frac{[M\Delta L]}{[\Delta L^2 T]}=\frac{[M]}{[L T]};$$

or, in C.G.S. units,

$$\frac{1\text{ gm.}}{1\text{ cm. }1\text{ sec.}}.$$

The conductivity of a substance is found in general to diminish with increasing temperature. In some experiments of Principal Forbes upon a square iron bar measuring $1\frac{1}{4}$ inches on a side, the conductivity was found to decrease from 0.207 at 0° C. to 0.124 at 275°. The thermal properties of metals in this and several other respects are remarkably like their electrical properties, as will be more fully shown hereafter.

205. Good and Bad Conductors.—A certain knowledge of the conductivity of a body at moderate temperatures may be obtained by means of the temperature sense alone. It is well known that a piece of metal feels much colder than a piece of cloth at the same temperature. The explanation is that the metal conducts the heat away from the hand more rapidly than the cloth does. The low conductivity of wool makes it a suitable fabric for the protection of the body in winter, since its heat will be carried off slowly, even when the outer surface is kept at a low temperature. Cotton and linen, which are better conductors, are for this reason more suitable for hot weather.

A rough comparison of the conducting powers of metals may be made by twisting together two or more wires, and attaching to them a series of balls by means of wax, as in Fig. 170. If a lamp be applied at one end, the wax will be melted as the heat flows along the wires, and the balls fall off one by one. The better conductor will ultimately melt the wax at the greater distance from the source of heat. The conductivities, as defined by equation 2, will be very nearly as the squares of these distances. It is, however, to be

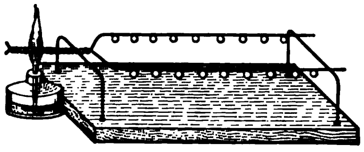


Fig. 170.

noticed that the first ball does not necessarily drop soonest on the bar from which the most balls are finally melted. The velocity of the temperature wave will evidently be greater in proportion as the specific heat is smaller. The property upon which the rate of change of temperature during the variable stage depends has been called the *diffusivity*, and is defined by

$$\kappa = \frac{k}{s}, \tag{3}$$

where s is the specific heat and k the conductivity.

TABLE OF THERMAL CONDUCTIVITIES OF SOLIDS.

SUBSTANCE.	TEMPERATURE C.°	CONDUCTIVITY. Unit $\frac{1\text{ gm.}}{1\text{ cm. }1\text{ sec.}}$.
Aluminum	0° to 100°	0.3435 to 0.3619
Antimony	0° to 100°	0.0442 to 0.0396
Bismuth	0° to 100°	0.0177 to 0.0164
Brass	0° to 100°	0.2041 to 0.2540
Copper	15°	0.713 to 0.9996
Flannel	− 10°	0.0000355
Glass	. . .	0.0018 to 0.0021
Iron	15°	0.149 to 0.201
Lead	0° to 100°	0.0836 to 0.0764
Marble	. . .	0.0012 to 0.0018
Mercury	0° to 50°	0.0148 to 0.0189
Phosphor Bronze	15°	0.4152
Silver	0°	1.0960
Snow	− 10°	0.00072
Tin	0° to 100°	0.1528 to 0.1423
Wood	. . .	0.00026 to 0.00059
Writing Paper	. . .	0.00012
Zinc	0°	0.3056
Zinc	15°	0.2545

206. Trevelyan Experiment. — The thermal properties of metals are exhibited in a somewhat unusual way in an occurrence first observed by Trevelyan, when a hot soldering iron was laid against a piece of lead. The experiment is best exhibited by means of the apparatus shown in Fig. 171.

The piece *A*, called the rocker, is made of brass, and grooved on one face so as to leave two projecting edges, shown at *A′*. If it is raised to a temperature of about 200° C. and laid on a bright cylinder of lead, *B′*, in the position shown, it will continue to oscillate back and forth from one

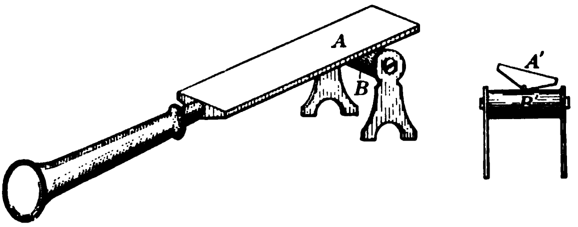


FIG. 171.

edge to the other for a long time, emitting a definite musical note. The explanation is that when one of the hot edges meets the surface of the lead, which has a large coefficient of expansion and a relatively small conductivity, a lump is raised with sufficient rapidity to tilt the rocker over upon the other edge, whence at once, in a similar manner, it is thrown back. The continual repetition of this process gives rise to a series of waves in the air, which produce the sensation of sound.

207. Davy's Safety Lamp. — If a piece of wire gauze be lowered over a gas flame, the flame will not pass through the meshes. Similarly, if the gas be first allowed to pass through the gauze and is then lighted from above, the flame

will remain entirely on the upper side. The explanation is that the heat is conducted away by the metal wires so that the temperature of the gas on the side opposite to the flame does not rise high enough for ignition. This principle was utilized by Davy in the construction of a safety lamp designed to prevent the explosion of fire-damp, a dangerous mixture of gases which often collects in coal mines.

By enclosing the miner's lamp in a cage of wire gauze, only so much of the explosive gas as passes through the gauze will burn, while the large body cannot ignite.

208. Conduction in Crystalline Media. — In anisotropic bodies the conductivity is different in those directions in which its other properties vary. If, for instance, a slice be cut from a piece of calcite parallel to the crystallographic axis, the varying conductivity may be exhibited by covering the surface with paraffine and inserting a heated rod in a hole previously bored through the center. The paraffine will be melted for a distance about the hole (Fig. 172) to a line which marks the temperature of fusion of paraffine. The form of the isothermal, thus indicated, will be an ellipse.

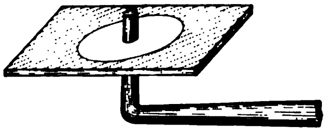


Fig. 172.

209. Conductivity of Liquids. — The difficulties in the way of the exact determination of the conductivity of liquids are considerable. If the lower strata are heated, the change in density sets up convection currents which entirely mask the property sought. But even if these are eliminated, it still is impossible to tell how much of the heat may have been transmitted by molecular diffusion. The following table shows the observed values of the conductivity of certain liquids in C.G.S. units.

SUBSTANCE.	TEMP.	CONDUCTIVITY.
Alcohol	13°	0.000545
Bisulphide of Carbou . .	"	0.000266
Ether	"	0.000378
Glycerine	"	0.000637
Oil of Turpentine . . .	"	0.000325
Water	18°	0.001245 to 0.001358

210. Conductivity of Gases. — In the case of gases, direct radiation increases the uncertainty which attaches to the determination of the thermal conductivity. Experiment indicates that its value is small. Poor conductors, such as felt, fur, or down, owe the property to the presence of interstices filled with air; for when these materials are compressed, to reduce the air cavities, they become much better conductors. The observed values for a few gases are as follows:

SUBSTANCE AT 7.56°.	CONDUCTIVITY.
Air	0.0000516
Carbon Dioxide	0.0000273
Hydrogen	0.00033
Marsh Gas	0.000065

The high value found for hydrogen is consistent with a number of other properties common to the elements called metals.

211. Cooling by Radiation. — When a hot body is suspended by a poor conductor, it is found to lose heat rapidly by radiation, at a rate depending on the nature of the body and the condition of its surface, being notably greater for a blackened than for a polished one.

The first experiments on the rate of cooling of a body seem to have been made by Newton, who expressed his conclusion in the statement that the rate of cooling was proportional to the difference of temperature between the body and the enclosing chamber.

If R be the rate of loss of heat, and t_1, t_2 the respective temperatures of the radiating chamber, Newton's Law may be expressed

$$R = A\,(t_1 - t_2). \tag{4}$$

This formula may be taken to represent the facts fairly well when $t_1 - t_2$ does not exceed a few degrees, but the departure from the truth is very noticeable when the difference of temperature is as much as 40° C.

The matter was later investigated by Dulong and Petit in an elaborate series of experiments on the cooling of thermometers in an exhausted chamber. As the result of their study they proposed the formula

$$R = k\;(a^{t_1} - a^{t_2}), \tag{5}$$

where k is a constant depending only on the nature of the material and the surface, and a is a number having the value 1.0077 if t is measured on the Centigrade scale. This law of Dulong and Petit has been accepted as expressing with approximate accuracy the rate of cooling within the range of their experiments, which was from 20° to 240° Centigrade. These experiments, which had been conducted with the utmost care, were reviewed in 1879 by Stefan, who found that the formula

$$R = c\,(t_1^4 - t_2^4) \tag{6}$$

fitted the observations more closely than the one given by the original investigators. Stefan's formula has since been deduced from theoretical considerations by both Boltzman and Galitzine.

212. Definition of Emissivity. — The *emissivity* of a surface is defined as the quantity of heat lost per second, per square centimeter, per degree difference of temperature be-

tween the body and the walls of its enclosure under standard conditions, *e.g.* immersion in air at atmospheric pressure. Determinations of emissivity, undertaken by M'Farland and by Bottomley, do not appear to support Stefan's Law.

213. Calorimetry by the Method of Cooling. — A method of calorimetry depending on the rate of cooling may be conveniently used in the case of liquids, especially with those which are obtainable only in small quantities. The substance to be examined is heated and placed in a thin copper vessel having a blackened surface, and supported without direct contact within a larger vessel, which is kept at a constant temperature by means of a large bath of water. The time occupied by the smaller vessel in cooling, through a definite range, is observed and noted. The liquid is then removed and water at the same initial temperature substituted. As the external surface of the vessel is the same in both cases, the rate of cooling will be the same as before. Hence if the time of cooling through a given range is observed, the quantity of heat lost in each case may be taken as proportional to the time. Thus, let T be the time it takes the substance to cool through a range $\theta_1 - \theta_2$, and T' the time occupied by the water in cooling through the same difference of temperature. Also,

Let m = mass of the liquid,
m' = " " " water,
s = specific heat of the liquid,
s' = " " " " water.

Then

$$\frac{ms}{m's'} = \frac{T}{T'},$$

or

$$s = \frac{Tm'}{T'm} s'.$$

214. Absorption. — It has already been remarked in Art. 202 that a body losing heat by radiation is a source of disturbance, giving rise to a train of waves which are propagated with the velocity of light through space not filled with gross matter. Since, in fact, these waves differ in no respect from those which produce the sensation of light, except that their wave-length may be greater, the phenomena and laws of their propagation may best be studied under the general subject of Light. These long waves are sometimes spoken of as *radiant heat*, but they are not heat in any strict sense of the term while they are in process of transmission; but if they fall upon some substance which is not able to pass them on unimpaired, they are transformed into heat, and the process of the change is called *absorption*.

In order to make clear what occurs in this case, it will be necessary to borrow an illustration from an analogous phenomenon in sound, known as resonance. When an elastic body, such as a column of air or a tuning fork, is capable of vibrating in one particular period, it will not be set into sensible vibration unless it receives a disturbance timed to that period; so that if several systems of waves of various periods fall upon a tuning fork, those waves whose periods are different from that of the fork pass on unaltered, but that system which is identical in period with the fork is absorbed, and the fork may be heard to emit the note peculiar to itself.

If the various systems of waves radiated by a heated substance fall on another body, those waves which it would emit when heated are absorbed, but most of the others are transmitted. The body is said to be *athermous* to the waves absorbed, but *diathermous* to those transmitted.

The corresponding terms applied to light waves are *opaque* and *transparent*.

It may be mentioned that rock salt is one of the most diathermous bodies yet examined, but further discussion of this subject will not be presented at this point, since many of the instruments used to measure radiation make use of electrical principles not yet explained.

215. Prévost's Theory of Exchanges. — When an isolated system of bodies is left to itself, it is known that the members of this system will ultimately, as a result of conduction or radiation, reach a common temperature. The question, whether under these circumstances the radiation has ceased or not, was first answered by Prévost in the negative. If, he reasoned, a cold body were to be placed in the midst of the system, it would immediately begin to grow warm by the radiations received; or if one of the bodies of the system were removed and placed among colder ones, this body would begin to lose heat. Now, a cold body has not the power of acting on another at a distance so as to make it begin to emit radiations. Therefore, a body is always emitting heat at a rate depending on its temperature, regardless of the presence of other bodies, and thermal equilibrium is maintained in the system considered only by each body receiving heat from the others at the same rate that it loses it by radiation. This theory of exchanges, as it is called, was implicitly assumed in the statement of Dulong and Petit's Law in Art. 211.

216. Effect of Radiation on Thermometers. — It follows from the considerations just explained that the reading of a thermometer may be greatly affected by the radiation to or from surrounding bodies, and that in order to obtain the true temperature of the air out of doors, special precautions are necessary. If the bulb be coated with polished silver, the effects of radiation may be greatly lessened, for the absorp-

tion of such a surface is only $\frac{1}{40}$ of that of lampblack. A better arrangement, however, is that suggested by Joule, namely, to surround the thermometer by a long vertical copper tube, open at the top but closed by a cap at the lower end. If the tube and the thermometer are at the same temperature, the radiation between them will have no effect on the reading. But there may still be some discrepancy between the temperature of the air in the tube and that outside. In that case convection currents either up or down the tube would arise on removing the cap. To detect the presence of such currents a wire helix is suspended within the tube in such a way as to indicate their direction. The copper tube is further supplied with a jacket through which warm or cold water may be passed until the column of air is in equilibrium. If the thermometer then shows a stationary reading, it may be accepted as the temperature of the air.

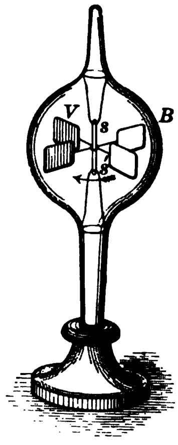


FIG. 173.

217. Crookes' Radiometer. — An interesting phenomenon attending radiation and absorption in an attenuated gas was discovered by Crookes in 1873. It may be exhibited by means of the instrument shown in Fig. 173, called a *radiometer*.

B is a bulb containing gas at a low pressure, say something like a centimeter of mercury. Attached to a spindle, *ss'*, and revolving with it are four vanes, *V*, of mica, blackened on one side and silvered on the other. When this apparatus is placed in the sunshine, or where the radiation from a hot body falls upon it, the vanes begin to revolve in such a direction that the blackened faces move away from the source of heat and the polished ones

toward it. The explanation is to be found in the fact that the blackened surface absorbs more heat than the bright one and acquires a higher temperature. In consequence the molecules which strike the vane bound off from this side with a greater velocity than from the other, at the same time giving the vane an impulse in the opposite direction. If the gas is too dense or too rare, the vane will receive no motion. When the instrument is placed before a very cold body, the radiation from the blackened face is more rapid, its temperature falls, and the vane will rotate in the opposite direction.

218. Dew. — When small bodies on the surface of the earth are exposed to a clear sky after sunset, the loss of heat by radiation soon cools them below the temperature of saturation of the surrounding air. The moisture which is then deposited is called dew. Since, however, the temperature cannot fall lower than the dew point until all the moisture has been abstracted from the atmosphere, it is evident that the presence of aqueous vapor in the air really serves to prevent the freezing and destruction of plant life during the night, provided the dew point is above 0° C. Any interposed body, such as the clouds, by the theory of exchanges, returns a part of the heat lost by radiation and greatly diminishes the deposition of dew. The continual stirring of the air on windy nights is also unfavorable to the formation of dew, since a fresh supply of heat is thus brought to the radiating body.

EXAMPLES.

1. A tank 6 meters long and 4 meters wide is filled with water which is covered with a layer of ice 6.6 cm. thick. If the temperature of the air is -11 , and the coefficient of conduction of ice be taken as 0.0023 gm. / cm. sec., how much heat will be transmitted through the ice per hour? *Ans.* $3.3(10)^6$ cal.

2. The top of a steam chest containing steam at atmospheric pressure consists of a slab of stone 61 cm. long, 49 cm. broad, and 9.9 cm. thick. The top being covered with ice, it was found that 4.8 kilos of ice were melted in 29 min. What is the conductivity of the stone? *Ans.* 0.0072 gm. / cm. sec.

3. Water is boiled at atmospheric pressure in an iron vessel having a heating surface of 2.34 square meters and a thickness of 1.16 cm. How much water will be evaporated per hour if the surface exposed to the fire is kept at 280° and the conductivity of iron be assumed as 0.164 gm. / cm. sec. *Ans.* $4(10)^6$ gms.

4. How much heat would be lost per square decimeter per minute by a man clothed in a fabric 0.3 cm. thick, having a conductivity $1.22(10)^{-4}$ gms. / cm. sec., if the temperature of the air is 5° and the temperature of the body is 30°. *Ans.* 60 cal.

CHAPTER XIV.

THERMODYNAMICS.

219. Mechanical Production of Heat. — Of the discovery that heat could be produced by mechanical means nothing is known, but the knowledge of the fact must have been in the possession of the race from the earliest times. Examples of such production are afforded by many familiar operations. For instance, the rubbing of a piece of steel against a rapidly revolving grindstone generates sufficient heat to raise the abraded particles of the metal to the temperature of ignition. The friction of a match against a rough surface likewise develops heat enough to ignite the phosphorus. A piece of lead may be rendered quite hot by simply striking it with a hammer, and the temperature of the water below a waterfall is raised an appreciable amount by the impact at the end of the fall. In the case of machinery which is run at a high speed, it is, in fact, very difficult to prevent the generation of a prejudicial quantity of heat at the journals.

In all the examples mentioned the production of heat involves the expenditure of a certain amount of work obtained either from the action of some external force or from the diminution of the kinetic energy by impact.

220. Fire Syringe. — Another interesting illustration of the generation of heat by the expenditure of work is afforded by the fire syringe. This instrument consists of a stout glass tube, *A* (Fig. 174), furnished with a close-fitting piston, *B*. A small quantity of carbon disulphide is introduced into the tube, filling the chamber with a mixture of

air and vapor. If the piston be suddenly pushed down, sufficient heat may be developed by the compression to ignite the vapor, producing a bright flash. Likewise, when a gas is allowed to expand, doing work on outside bodies, its temperature falls on account of the disappearance of heat.

A

FIG. 174.

221. Theories Concerning the Nature of Heat. — Two rival theories concerning the nature of heat existed side by side from the time of the ancient Greeks until the middle of the present century. According to one, heat was a highly elastic and self-repellent fluid termed *caloric*, which pervaded the interstices of all bodies. In the other doctrine, heat was ascribed to the rapid vibration of the molecules of a body. The first of these hypotheses was the generally accepted doctrine until shortly after the beginning of the present century. The history of its final overthrow marked a great advance in science, for so long as heat was accorded a material existence, the establishment of the doctrine of conservation of energy was impossible.

222. The Caloric Theory. — The fundamental property of heat demanded by the fluid theory was that it be indestructible and uncreatable. When it was added to bodies they became warmer, and when it was abstracted they grew cooler. To account for the different thermal capacities of substances it was supposed that they had different attractions for caloric, and that some would absorb a greater quantity than others in rising through the same difference of temperature. The expansion of bodies was regarded as the natural consequence of the absorption of this self-repellent fluid. To account for

the disappearance of heat at the change of state, Black supposed that caloric could exist not only in the free state, but that in such a case as the melting of ice, it became inactive or so hidden that it could not be detected by the thermometer, whence he termed it *latent*, a name which is still retained. Conduction was explained by supposing that the fluid, caloric, in consequence of the mutual repulsion of its parts, flowed from high temperatures to low, just as water runs from higher levels to lower.

To explain the heating of bodies by percussion the calorists assumed that some of the fluid was squeezed out. In the case of friction it was supposed that the portion of the body which was reduced to powder by the abrasion of the surface had a less capacity for heat, and in this treatment gave up some of its caloric. This important postulate might easily have been submitted to experiment, but the calorists allowed their theory to rest upon bare speculation.

223. Experiments of Rumford. — The first experiments for the purpose of determinating the nature of heat were undertaken by Count Rumford at the close of the last century. While engaged in boring a brass cannon he was much impressed by the amount of heat developed in the process, and, seeking to obtain exact knowledge of the matter, he prepared a special experiment, in which a blunt steel borer was pressed against the bottom of a brass cylinder provided with a thermometer to indicate the temperature. After 960 revolutions it was observed that the temperature had risen 70° F., though the mass of metal abraded was only $\frac{1}{950}$ of the whole. It seemed to Rumford improbable that so large a quantity of heat should be given up by the diminution of the thermal capacity of so small an amount of metal. Further experiments on massive and pulverized brass show-

ing that the specific heat in each case was sensibly the same, Rumford rejected the hypothesis that heat was liberated by the change in the thermal capacity of bodies incident to their change of form.

But if heat is a fluid, it is necessary to admit that an unlimited quantity can be generated by an insulated system of bodies, for in the experiment there had appeared no sign of diminution or exhaustion. Regarding such a conclusion as inadmissible, Rumford was led to the view "that the only thing capable of being excited and communicated in these experiments is *motion*."

224. Davy's Experiment. — The crucial test of the caloric theory was devised by Davy in the following year, 1799. By simply rubbing two blocks of ice together he showed that they could be entirely converted into water. Now, since this change of state admittedly involves the absorption of considerable heat, the hypothesis that caloric is released by the operation is clearly disproved.

225. Mechanical Equivalent of Heat. — The now universally adopted theory that *heat is the kinetic energy due to the irregular motion of the molecules of a body* was received with small favor during the first quarter of the century; from that time on, however, it gained increasing support, and by 1840 its acceptance had become general. The chief contributor to this result was Dr. Joule of Manchester, who, assured that heat was a form of energy, undertook in the year mentioned to find out the numerical relation between the units of heat and work. His measurements were made upon the elevation of temperature of a known mass of water when churned by the fall of a weight. The apparatus consisted essentially of a copper vessel, *AB* (Fig. 175), containing a paddle attached to a vertical roller, *f*, which was made to revolve

by the descent of the weights E, E. The paddle itself (Fig. 176) was made of brass and consisted of eight sets of arms, a, which worked between stationary vanes, b, in the vessel,

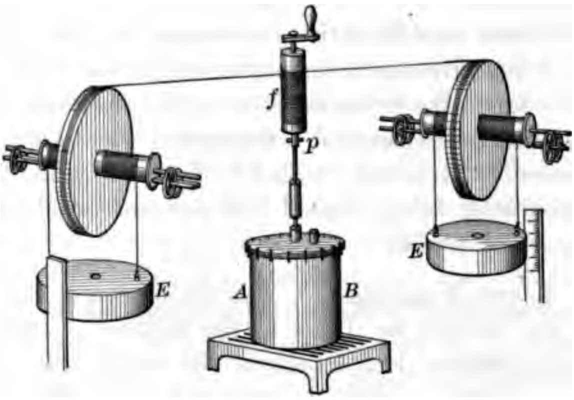


FIG. 175.

so that the water was prevented from revolving bodily with the paddle. The roller f could be detached from the paddle at pleasure by removing a pin, p, and the weights wound up without agitating the water. Joule's experiments extended over a number of years, and showed, when corrected as nearly as possible for such sources of error as could not be eliminated, that 772 foot-pounds of work at Manchester $\left(g = 981 \frac{\text{cm.}}{\text{sec.}^2}\right)$ would raise the temperature of 1 pound of water at 60° F. through 1° F. This equivalent is denoted by J. Observing that its dimensions are those of work per unit mass, per degree,

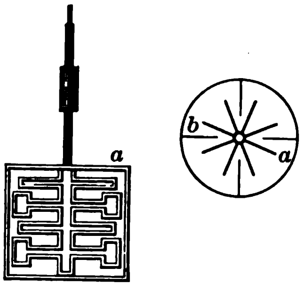


FIG. 176.

$$J = 772 \frac{\text{foot-pounds}}{\text{pound } 1^\circ \text{ F.}} = 4.159(10)^7 \frac{\text{ergs}}{\text{gm. } 1^\circ \text{ C.}}.$$

226. Rowland's Determination of J. — The determination of Joule remained the adopted value of the mechanical equivalent until 1879, when Rowland undertook the most careful and complete study of this constant which has ever been made. The method was similar to that employed by Joule, but the details were improved in several particulars. The paddles, more elaborate in design, were turned by a steam engine, thus securing a more rapid change of temperature, 35° C. per hour, instead of 0.62° C., obtained by Joule.

The mercurial thermometer, which might introduce errors of one or two per cent, was replaced by the air thermometer. After all corrections had been applied, Rowland's experiments showed

$$J = 4.187(10)^7 \frac{\text{ergs}}{\text{gm. } 1^\circ \text{ C.}} \text{ at } 15.8^\circ \text{ C.},$$

as compared with $4.159(10)^7$, as found by Joule at the same temperature. The principal part of this difference is to be ascribed to want of agreement between the mercurial and the air thermometers. It was by this series of experiments that Rowland established the fact that the specific heat of water is not constant. (See Art. 153.)

227. Other Determinations of J. — The value of the mechanical equivalent of heat has been determined by other experimenters in a variety of ways both direct and indirect. The most reliable of the latter methods is to measure the heating of a wire by the passage of a known electric current. There is a slight discrepancy, as may be seen from the following table, between Joule's and the electrical method, which has not yet been accounted for.

MECHANICAL EQUIVALENT OF HEAT

Author	[illegible]	[illegible]	Remarks
[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	Joule [illegible]
. .	[illegible]	[illegible]	Rowland
. .	[illegible]	[illegible]	Miculescu
[illegible]	[illegible]	[illegible]	Griffiths
.	[illegible]	[illegible]	Schuster

228. The Two Laws of Thermodynamics. — The processes of reasoning employed in the discussion of problems in heat are founded on two fundamental principles relating to the conversion of heat into work, and known as the first and second laws of thermodynamics.

The first law asserts that when heat is transformed into work, or work into heat, the quantity of work is equivalent to the quantity of heat. This relation, as has been seen, was first established by Joule, and may be written

$$(1) \qquad W = JH = Q,$$

where W is the work measured in ergs, J the mechanical equivalent of heat, H the quantity of heat measured in calories, and Q the same quantity measured in ergs. In other words, the first law declares the identity of heat with energy and brings it within the general law of conservation of energy.

The second law of thermodynamics asserts that heat cannot of itself pass from a cold to a hot body, or more formally, it is impossible, by means of inanimate material agency, to obtain work from any portion of matter by cooling it below the temperature of the coldest of surrounding objects.

There is no a priori reason for denying that work can be derived by using up the heat of a single body. For, if the ultimate material particles by the motion of which the phenomenon of heat is ascribed, be supposed large enough to be guided and controlled, it would not be impossible to prepare a machine which should by a suitable train of mechanism transfer the energy of these particles to another body and convert it into work. The second law thus reduces to the assertion that the molecules of a body are so small that it is beyond our power to manipulate them.

229. Adiabatic Expansion. — A body is said to undergo an adiabatic change when its condition is altered without its gaining or losing heat to other bodies. In the case of the adiabatic expansion of a gas the relation between the pressure and the volume of a gas may be expressed by

(2) $$pv^{\gamma} = \text{constant},$$

where γ is the ratio of the specific heats. See Art. 24[illegible]. If this equation be plotted on the PV diagram the adiabatic line AA (Fig. 177) will be more inclined to the horizontal than the isothermal II, or, what amounts to the same thing, if a gas be compressed from v_1 to v_2 without permitting gain or loss of heat, the pressure rises faster than when the temperature remains constant. Since for a given change in volume the greater change in pressure corresponds to a

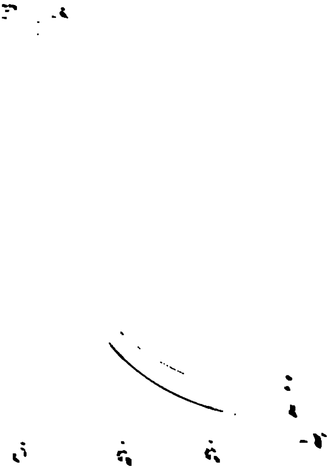


Fig. 177.

greater change in temperature, it follows that the gas is heated by compression and cooled by expansion.

230. Work Done in Compressing a Fluid. — Suppose a cylinder to be filled with a fluid and fitted with a freely moving piston. In order to calculate the work done in moving the piston a distance, x, call the area of the piston A, and the average pressure during the motion p, then the work done will be

(3) $$W = pA \cdot x;$$

but Ax equals the change in volume, say u.

Substituting,

(4) $$W = pu,$$

a result which is independent of the area of the piston and of the volume and form of the vessel.

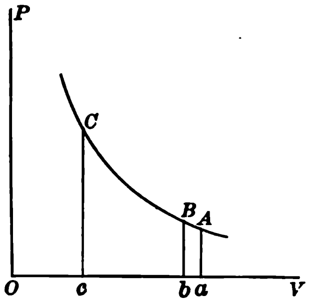


FIG. 178.

The work done in changing the volume of a fluid may be conveniently represented on the pressure-volume diagram as follows. Suppose the initial state of the body is represented by the point A (Fig. 178), and that its condition is altered in any manner until it reaches the state denoted by the point C. If B represent an intermediate stage very near to A, the average pressure for the small change is given by

$$\tfrac{1}{2}(Bb + Aa)$$

and the diminution of volume by ba. Hence, the work done will be given by

(5) $$W = \tfrac{1}{2}(Aa + Bb)ab = \text{area } bBAa.$$

Now, since the whole change from A to C may be regarded as made up of the sum of such small changes as that from A to B, the work done in changing the body from the state A to the state C, along the path BC, is proportional to the area $bBCc$. It will be convenient to observe a convention of signs, so that this area shall be described by a negative rotation when work is done on the body, and by a positive rotation when the body does work. Thus, $CcaA$ will represent work done on the body, and $CAac$ work done by the body.

231. Carnot's Cycle. — One of the earliest investigators of the manner in which work is produced by heat was Sadi

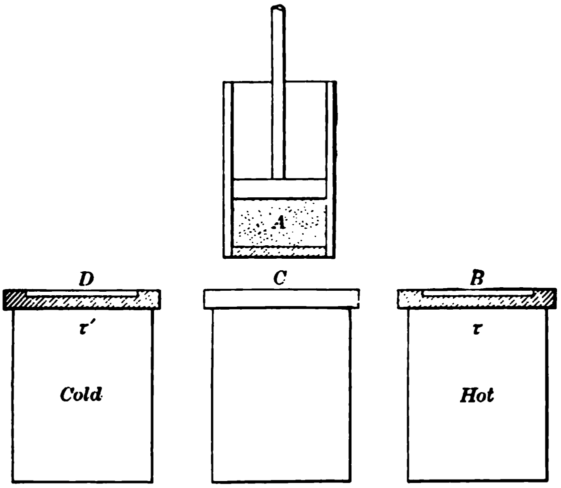


FIG. 179.

Carnot. His most noteworthy contribution to this branch of science was a new method of reasoning, in which a body was made to pass through a cycle of operations.

For simplicity suppose that the working substance is a gas, A, enclosed in a cylinder (Fig. 179) furnished with a piston,

which, with the side walls of the cylinder, is absolutely impermeable to heat, but that the bottom of the cylinder is a perfect conductor. Suppose also that B and D are two bodies maintained constantly at the respective temperatures τ and τ', and that C is a non-conducting stand on which the cylinder may be placed. The four distinct operations of Carnot's cycle may be thus described:

First Operation. Let the cylinder and contents at the temperature τ' be placed on the stand C, and the gas compressed without escape of heat till its temperature rises to τ. If the initial state of the gas be represented by E (Fig. 180), the change during the first operation will be represented by the adiabatic line EF, and the work done by the area $EFfe$.

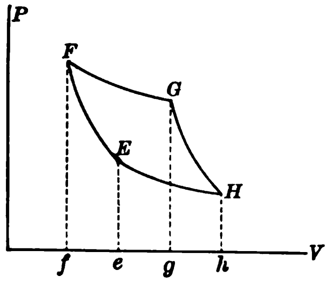


Fig. 180.

Second Operation. Let the cylinder be transferred to B, and the piston allowed to rise gradually. During this process a quantity of heat, Q, will flow in through the perfectly conducting bottom, and maintain the temperature constantly at τ. This change is represented by the isothermal FG (Fig. 180), and the work done by the gas is represented by the area $FGgf$.

Third Operation. Let the cylinder be placed on the insulating stand, and the gas allowed to expand until the temperature falls to that of the cold body B, namely, τ'. This change is represented by the adiabatic GH, and the work done by $GHhg$.

Fourth Operation. Let the cylinder be placed on D, and the piston pushed down till it reaches the starting point. The heat Q', which is developed by this process, is absorbed

by D, and the temperature of the gas remains constantly τ'. This change is represented by the isothermal HE, and the work done by $HEeh$. The body has now passed through a complete cycle of operations, and reached its initial condition. The total work done during the cycle will be found by subtracting the area $HEFfh$, which represents the negative work, from the area $FGHhf$, which represents the positive work. This difference, $EFGH$, represents the work done by the body. As the working substance is left in precisely the same condition as at first, the physical results of the process are: 1°, the removal of a quantity of heat, Q, at a temperature, τ, from B; 2°, the performance of a quantity of work represented by the area $EFGH$; and 3°, the communication of a quantity of heat, Q', at a temperature, τ', to the body D.

By the first law of the thermodynamics the relation between these quantities, measured in ergs, is

$$Q - Q' = w. \tag{6}$$

232. Reversible Cycle. — If the working substance may be made to pass through the successive stages of the cycle in the reverse order, the process is said to be *reversible.* It is evident, on examination of the cycle just described, that if the engine were run backwards, the physical results of the operation would be a quantity of heat, Q', taken from D; a quantity of work, represented by the area $EHGF$, done on the substance, and a quantity of heat, Q, delivered to B. In this case heat has been transferred from a cold body to a hot one, but only at the expense of mechanical work. It thus appears that the transfer of heat from one body to another, by the highly artificial process described, is a completely reversible process.

An example of an irreversible process is seen in the transfer of heat by conduction, for when a hot body is brought in contact with a cold one, heat passes of itself from the hot to the cold body, but never from the cold to the hot body.

233. Efficiency of an Engine. — The efficiency of an engine is defined as the ratio of the work done by the engine to the quantity of heat drawn from the source. If w is the work done, and q the quantity of heat received, both being measured in units of energy, then the efficiency E may be written

$$E = \frac{w}{q}. \tag{7}$$

234. Carnot's Theorem. — The following important principle was discovered by Carnot. If a reversible engine, working between the temperatures t and t', receive a quantity of heat, q, measured in ergs, at the higher temperature t, and produce a quantity of work, w, then the efficiency

$$\frac{w}{q} = E$$

is the greatest that can be obtained by any engine working between the given temperatures. For, suppose that the engine A_2 has a greater efficiency than that of a reversible engine A_1; and, further, suppose that A_2 takes a quantity of heat, Q, from a hot body, as B (Fig. 179), and gives up Q'_2 units of heat to the cold body D, doing an amount of work, W_2. Let A_2 be used to drive A_1 backwards, and suppose that the latter takes a quantity, Q'_1, from D, and yields a quantity, Q, to B, at the expenditure of the work W_1. The efficiency of A_2 is $\frac{W_2}{Q}$, and that of A_1 is $\frac{W_1}{Q}$. But by hypothesis

$$\frac{W_2}{Q} > \frac{W_1}{Q}, \text{ or } W_2 > W_1;$$

that is to say, the compound engine does an available amount of work, $W_2 - W_1$, by the expenditure of a quantity of heat, $Q'_1 - Q'_2$. For the source B is unaffected by the stroke, since A_2 takes Q units from it, and A_1 returns the same amount. Hence, if A_2 does more work than A_1, it must yield less heat to D than A_1 would under the same circumstances, that is to say, $Q'_2 < Q'_1$, so that

$$W_2 - W_1 = Q'_1 - Q'_2. \tag{8}$$

An amount of work is thus done by using up the heat of the colder body, and the process might be continued indefinitely. But this is in violation of the second law of thermodynamics; therefore, no engine can be constructed having a greater efficiency than the reversible engine.

It follows as a corollary that no reversible engine can have a greater efficiency than any other reversible engine, *i.e.* all reversible engines have the same efficiency. Hence the efficiency of a reversible engine is independent of the nature of the working substance, and consequently must depend alone on the temperatures between which the engine is worked.

235. The Steam Engine. — The most important method of transforming heat into work for commercial purposes is

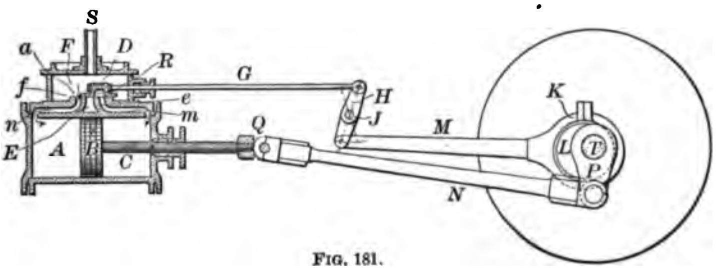


FIG. 181.

by means of the steam engine. A sketch of a simple type is shown in Fig. 181, with the parts lettered according to the annexed scheme :

A. Cylinder.
B. Piston.
C. Piston Rod.
m. *n*. Steam-ways.
E. Exhaust Port.
D. Slide Valve.
f. Valve Seat.
F. Steam Chest.
a. Steam Port.
G. Valve Stem.
H. Rocker.
J. Rocker Shaft.
L. Eccentric.
K. Eccentric Strap.
M. Eccentric Rod.
N. Connecting Rod.
P. Crank.
Q. Cross Head.

If steam be admitted to the chest *F* by the pipe *S*, it will pass through the open port *a*, force the piston *B* to the right, and impart a rotation to the shaft *T*. The slide valve *D* is, in consequence, moved to the left, closing the port *a* and allowing the steam in *A* to expand. When the piston has reached the end of its stroke, the valve has moved far enough to allow steam from the chest to enter the port *e* and force the piston to the left, while the steam in *A* escapes through *a*, *R*, and *E*.

236. Indicator Diagram. — A steam engine may be made, by means of a proper mechanism, to trace automatically the relation between the volume and the pressure of the steam in the cylinder during a complete stroke.

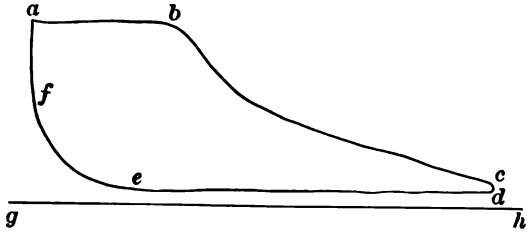


FIG. 182.

Such a pressure-volume diagram, technically known as an indicator card, is shown in Fig. 182, in which ordinates represent pressures, and the abscissas, volumes.

The record of the succession of changes which went on within the cylinder is to be interpreted as follows: From a to b steam was admitted at boiler pressure; from b, the point of cut-off, to c, the point of release, the steam expanded adiabatically; at d the pressure had fallen to its minimum value and remained sensibly constant until e was reached, when the exhaust port closed and the steam began to be compressed, as is indicated by the line ef. At f the steam port opened again and the pressure promptly rose to its original value.

The work done in this stroke may be calculated at once from the area $abcdef$, if the length of the stroke and the scale of the pressures are known.

The rate at which work is done by engines is measured in a unit called a *horse-power*, which is defined by

$$\text{One horse-power} = 33{,}000 \, \frac{\text{ft. pds.}}{\text{min.}}.$$

In the metric system $(10)^7 \dfrac{\text{ergs}}{\text{sec.}}$ is called a *watt.*

Accordingly $\qquad 1 \; h.p. = 746$ watts.

237. Expression for the Efficiency. — The fact that the efficiency of a reversible engine is a function of the temperatures alone may be concisely expressed by writing

$$E = \frac{W}{Q} = \phi(t, t'), \tag{9}$$

or if Q' is the quantity of heat returned to the condenser, by substituting the value of W,

$$\frac{Q - Q'}{Q} = \phi(t, t'), \quad \text{or} \tag{10}$$

(11) $$\frac{Q'}{Q} = 1 - \phi(t, t'),$$

and hence

(12) $$\frac{Q}{Q'} = \frac{1}{1 - \phi(t, t')} = \psi(t, t'), \text{ say,}$$

$\psi(t, t')$ being some other function of the temperatures. Now, referring to Fig. 180, it is evident that in passing from F to G, the quantity of work done, represented by $FGgf$, depends on the temperature t, the nature of the substance s, and the path FG, *i.e.* on the relation of p to v, and on nothing else. Therefore the quantity of heat received may be written

(13) $$Q = f(t, s, p, v),$$

and, similarly,

(14) $$Q' = f(t', s, p, v).$$

Dividing,

(15) $$\frac{Q}{Q'} = \frac{f(t, s, p, v)}{f(t', s, p, v)}.$$

But since by equation 12 $\frac{Q}{Q'}$ is a function of the temperatures alone, $f(t, s, p, v)$ must be of the form $F(s, p, v) \cdot f(t)$, so that

(16) $$\frac{Q}{Q'} = \frac{f(t)}{f(t')}.$$

Therefore since Q is greater than Q', and t greater than t', by the conditions of the problem $f(t)$ is an increasing function of t. Let this function for the temperature t be denoted by τ, then

(17) $$\frac{Q}{Q'} = \frac{\tau}{\tau'},$$

and

(18) $$\frac{Q - Q'}{Q} = \frac{\tau - \tau'}{\tau}.$$

The efficiency will then be completely determined as soon as the values of τ and τ' are found.

238. Thermodynamic Scale of Temperatures. — It was suggested by Kelvin, as early as 1848, that since the ratio $\frac{Q}{Q'}$ of the quantities of heat taken in and rejected between the temperatures t and t' depends on these temperatures alone, the numbers τ, τ' might be taken to represent the temperatures denoted by t and t' on the Centigrade scale, thus forming a new or thermodynamic scale. The thermodynamic scale leads at once to the notion of an absolute zero. For, suppose τ' chosen so that no heat is ejected by the engine; the efficiency would then be unity, and all the heat taken from the source would be turned into work.

But as no engine could be supposed to convert more heat into work than it received, it is impossible for τ' to be negative. Therefore zero is the smallest value it can have. Temperature reckoned from this zero is called absolute, and is independent of the properties of any substance. One thing still remains arbitrary on this scale, that is, the size of the degrees. This may be conveniently chosen so that there shall be one hundred degrees between the temperatures of boiling and of freezing water. If, now, a reversible engine be worked between the temperatures named, which may be denoted by τ_B and τ_F, and if the quantities of heat received and rejected be measured,

$$\text{(19)} \qquad \frac{Q - Q'}{Q} = \frac{\tau_B - \tau_F}{\tau_B} = \frac{100^\circ}{\tau_B},$$

from which the temperature of boiling water on the absolute scale may be found.

It is not necessary actually to try the experiment, for the work done by the expansion of a substance which obeys

Boyle's Law may be calculated by the methods of the infinitesimal calculus. Kelvin's thermodynamic scale has in this way been shown to be identical with that of a perfect gas thermometer.

239. Correction of the Zero on the Air Thermometer.— Since no body obeys Boyle's Law perfectly, the value of the absolute temperature of freezing water determined by experiment on gases requires a correction for the deviation from this law. A special examination of the properties of certain gases for the purpose of finding the amount of this correction was undertaken by Kelvin and Joule, making use of a method which is commonly known as the *porous plug experiment.*

The theory of the experiment may be explained by the aid of Fig. 183, which represents a tube stopped by a partition, A, in which is a hole filled with a plug of cotton wool.

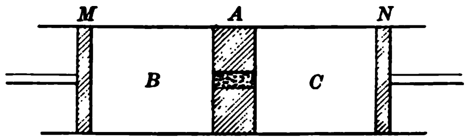


FIG. 183.

The chamber B is filled with a gas which is forced through the porous plug by means of a piston moved so as to maintain a constant pressure, p, on the left side of the partition. After passing through the plug, the gas in C is kept at a constant pressure, p', by means of the movable piston N.

Call the volume of unit mass of the gas on the respective sides of the partition v and v'. Then the work done by the piston M on the unit mass in moving up to A is pv, and, similarly, the work done by this amount of gas on the piston N is $p'v'$. If, now, the gas is of such nature that $p'v' > pv$, it might be expected that the temperature would fall, in consequence of expansion, through the plug; but if the product

pv is greater on the high-pressure side there would be a heating effect. Also, if there is any vestige of molecular attraction between the particles of the gas, simple rarefaction, without doing external work, should produce a cooling effect, so that any observed change of temperature on passing through the plug is to be regarded as the algebraic sum of effects arising from two different causes.

The results of the observations of Joule and Kelvin, made in the manner described, are shown in the following table:

Name of Gas.	Expansion at one Atmosphere between Freezing and Boiling Points (Regnault) α.	Mean Cooling Effect per Atmosphere.	Uncorrected Temperature of Melting Ice, $\frac{100}{\alpha}$.	Correction calculated from Cooling Effect.	Absolute Temperature of Melting Ice.
Hydrogen	0.36613	$-0.039°$	273.13°	$-0.13°$	273
Air	0.36706	$+0.208°$	272.44°	$+0.70°$	273.14
Carbonic Acid	0.37100	$+1.005°$	269.5°	$+4.4°$	273.9

It will be noticed that air and carbon dioxide were cooled by the expansion, as was also oxygen; but hydrogen was heated, thus showing that hydrogen stands out from the other gases in this respect, as it did in the experiments of Amagat. The experiments on air are regarded by Joule and Kelvin as the most trustworthy, so that 273.14° A. is to be accepted as the nearest approximation to the temperature of melting ice.

240. The Two Specific Heats of a Gas. — If it be assumed that no work is done in separating the molecules of a gas when it expands, the excess of the heat supplied at constant pressure over that at constant volume is exactly the work done in expansion. Call v_1 the initial and v_2 the final volume. Let m be the mass, p the constant pressure, and C_v

and C_p the two specific heats; then the work done in expansion is

(20) $$p(v_2 - v_1) = mR(t_2 - t_1).$$

The excess of heat necessary to change the temperature from t_1 to t_2 over that required at constant volume is

(21) $$m(C_p - C_v)(t_2 - t_1),$$

and its equivalent in units of work is

(22) $$Jm(C_p - C_v)(t_2 - t_1) = mR(t_2 - t_1);$$

whence

(23) $$C_p - C_v = \frac{R}{J}.$$

The ratio

(24) $$\frac{C_p}{C_v} = \gamma$$

may be determined from experiments on the velocity of sound in the gas (see Art. 494):

In the case of air

$$\gamma = 1.408,$$

$$R = 2.871(10)^6 \frac{\text{ergs}}{\text{gm. } 1^\circ \text{ C.}};$$

also $$J = 4.187(10)^7 \frac{\text{ergs}}{\text{gm. } 1^\circ \text{ C.}}.$$

Thus $$C_v = \frac{R}{J(\gamma - 1)} = 0.1681,$$

$$\frac{R}{J} = 0.0686,$$

$$C_p = 0.2367.$$

The value of this constant, found by Regnault in a direct determination, was

$$C_p = 0.2374.$$

241. Two Elasticities of a Gas. — The expression $\frac{\text{stress}}{\text{strain}}$, which was taken in Art. 66 as the definition of elasticity, will have two values according as the temperature remains constant, or as heat is prevented from entering or leaving the body. In the case of gases the expressions for these two elasticities, which will be denoted respectively by E_τ and E_η, have very simple forms.

Call the initial pressure of the gas P, and the corresponding volume V, and let v be the change in this volume due to a very small increase of pressure, p, at constant temperature, then

$$E_\tau = \frac{\text{stress}}{\text{strain}} = \frac{p}{\frac{v}{V}} = \frac{p}{v}V; \tag{25}$$

but by Boyle's Law

$$(P+p)(V-v) = PV; \tag{26}$$

that is,

$$pV = Pv + pv, \tag{27}$$

or

$$\frac{p}{v} = \frac{P}{V} + \frac{p}{V}, \tag{28}$$

which has the limit $\frac{P}{V}$ as p approaches zero.

Substituting in (25),

$$E_\tau = P. \tag{29}$$

To find the other elasticity, E_η, let τ_1, τ_2 (Fig. 184) be isothermals drawn on the pressure-volume diagram for a difference of temperature of 1°, and, similarly, η_1, η_2, two consecutive adiabatics. Then, in passing from the state η_1 to η_2, at the constant temperature τ_1, that is in the diagram, moving from O to M, the increase of pressure is represented by the

line LM, and the increase of volume by OL. Wherefore,

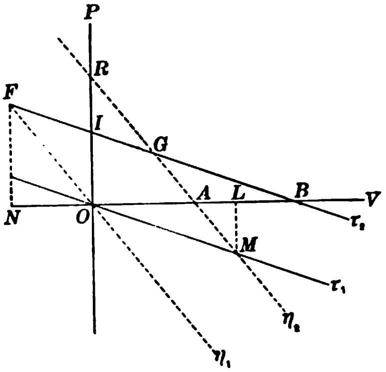


FIG. 184.

$$E_\tau = LM \div \frac{OL}{V} \tag{30}$$

$$= \frac{LM}{OL} \cdot V = \frac{OI}{OB} V.$$

Likewise, if the passage from the state O to the state F be without gain or loss of heat, the pressure changes by an amount represented by NF, and the volume by ON. Therefore,

$$E_\eta = NF \div \frac{ON}{V} = \frac{NF}{ON} V = \frac{OR}{OA} \cdot V. \tag{31}$$

If the mass of the gas considered be taken as a gram, the addition of a quantity of heat numerically equal to the specific heat at constant volume, C_v, would produce a rise of temperature of 1° and an increase of pressure represented by OI. Let OR be the increase of pressure by the addition of Q units of heat under constant volume. Then, as these changes are proportional to the quantities of heat which produced them,

$$\frac{C_v}{Q} = \frac{OI}{OR}. \tag{32}$$

Similarly, if C_p units of heat be added and the pressure be kept constant, the expansion per degree is represented by OB and the expansion produced by the addition of Q units is given by OA, the quantity of heat to pass from O to A being the same as from O to R, since the change RA requires no heat. Thus,

$$\frac{C_p}{Q} = \frac{OB}{OA}. \tag{33}$$

Dividing (33) by (32),

$$(34)\qquad \frac{C_p}{C_v}=\frac{\dfrac{OB}{OA}}{\dfrac{OI}{OR}}=\frac{V\cdot\dfrac{OR}{OA}}{V\cdot\dfrac{OI}{OB}}=\frac{E_\eta}{E_\tau}=\gamma,$$

so that, finally,

$$(35)\qquad E_\eta=\gamma E_\tau=\gamma P.$$

242. Lowering of the Freezing Point by Pressure. — The amount of work done by the unit mass of water when it freezes under a pressure, p, is

$$(36)\qquad W=p\,(v_i-v_w),$$

where v_i and v_w stand respectively for the volume of the ice and of the water.

If the ice be now melted in a vacuum, the substance will have passed through a complete cycle, and as it does no work in melting under this condition, equation 36 gives the total work. It is also evident that the process is a reversible one, for the water might be frozen in a vacuum and melted under pressure. The heat received in melting will be exactly L, the latent heat of water.

Thus, applying Carnot's theorem for the efficiency, equation 18 becomes

$$(37)\qquad \frac{W}{JL}=\frac{p\,(v_i-v_w)}{JL}=\frac{\tau-\tau'}{\tau},$$

or

$$(38)\qquad \tau-\tau'=\frac{p\,(v_i-v_w)}{JL}\,\tau.$$

Substituting the following numerical values :

$$p=1.013(10)^6\,\frac{\text{dynes}}{\text{cm.}^2}=1\text{ atmo.},$$

$$J = 4.186(10)^7 \frac{\text{ergs}}{1 \text{ gm. } 1^\circ \text{ C.}},$$
$$L = 79.25 \text{ gms. } 1^\circ \text{ C.},$$
$$\tau = 273^\circ \text{ C.},$$
$$v_i - v_w = 0.091 \text{ cu. cm.},$$

the lowering of the freezing point of water per atmosphere is found to be

$$\tau - \tau' = 0.007586^\circ \text{ C.}$$

Experiments conducted by James Thomson showed

$$\tau - \tau' = 0.0075^\circ \text{ C.}$$

243. Entropy. — Suppose that a substance is carried through any series of changes by a reversible process so as to return to the initial state, and let the series be represented by the closed curve of Fig. 185. Divide the figure

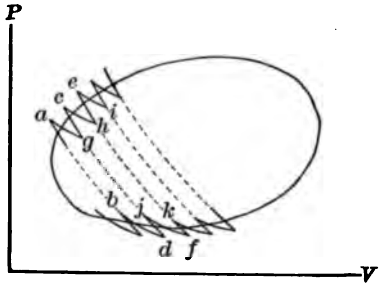


FIG. 185.

up into a series of little cycles by drawing adiabatics *ab*, *cd*, *ef*, etc., and isothermals *ag*, *bd*, *ch*, *jf*, etc., corresponding to the temperatures τ_1, τ_2, τ_3, etc. Suppose that the quantity of heat q_1 is received by the body in passing from *a* to *g*, and a quantity, q_2, given up in passing from *d* to *b*, and, similarly, q_3 to be received along *ch* and q_4 rejected along *fj*, and so on. For each little cycle the equations following will hold :

$$(39)\qquad \begin{cases} \dfrac{q_1}{\tau_1} = \dfrac{q_2}{\tau_2}, \\ \dfrac{q_3}{\tau_3} = \dfrac{q_4}{\tau_4}, \\ \cdots\cdots\cdot \\ \dfrac{q_{n-1}}{\tau_{n-1}} = \dfrac{q_n}{\tau_n}, \end{cases}$$

or, if quantities of heat received be called positive and those rejected negative,

$$(40)\qquad \begin{cases} \dfrac{q_1}{\tau_1} + \dfrac{q_2}{\tau_2} = 0, \\ \cdots\cdots\cdots\cdot \\ \dfrac{q_{n-1}}{\tau_{n-1}} + \dfrac{q_n}{\tau_n} = 0. \end{cases}$$

By addition,

$$(41)\qquad \frac{q_1}{\tau_1} + \frac{q_2}{\tau_2} + \cdots \frac{q_n}{\tau_n} = \Sigma \frac{q}{\tau} = 0,$$

that is to say, if the body be made to change its state by the addition of heat along the irregular path *aych* $\cdots$, etc., and by the abstraction of heat along *κfjbd* $\cdots$, etc., the sum of each of the quantities q divided by the temperature at which it is received or rejected is zero when the body returns to its initial condition. Now, by taking the divisions of the broken path smaller and smaller, this path may be made to approach without limit the closed, curved path in which the heat is received and rejected in a perfectly continuous manner, and the limit of the sum $\Sigma \frac{q}{\tau}$ is zero. It is also evident that this relation holds without respect to the form of the path as long as it is a reversible one and the body is brought back to its initial state. This statement is equivalent to saying that if the body be changed from the state A to the state B by any reversible path, the limit which the sum of

such quantities as $\frac{q}{\tau}$, taken from the state A to the state B, approaches is a constant quantity depending only on the pressure, volume, and temperature at the state A and at the state B, and not at all upon the intermediate stages.

In mathematical symbols this may be expressed

$$(42) \qquad \underset{q=0}{\Sigma_A^B} \frac{q}{\tau} = \phi_B(p, v, \tau) - \phi_A(p, v, \tau) = \eta_B - \eta_A.$$

The final conclusion, then, is this: that there is a distinct and measurable physical property of a body, which may be calculated from its pressure, volume, and temperature, and which is characterized by the peculiarity that it increases or diminishes as heat enters or leaves the body, but remains constant when there is no communication of heat. This quantity, defined mathematically by equation 42, and physically by the preceding paragraph, is called the *entropy*. It will be denoted by η. The importance of this function is greatest in those investigations which introduce the methods of the Calculus, but some further notion of its significance may be obtained by considering the transformations of a body in two simple cases.

Case 1. *Adiabatic Change.*

If the state of a body be changed without communication of heat, the change of entropy

$$(43) \qquad \eta_2 - \eta_1 = \Sigma_1^2 \frac{q}{\tau} \text{ is zero,}$$

because q is zero. An adiabatic line is therefore one of constant entropy and may be called an *isentropic*.

Case 2. *Isothermal Change.*

If the state of a body be changed under constant temperature, τ may be placed outside the sign of summation, since it is constant; that is,

$$(44) \qquad \eta_2 - \eta_1 = \Sigma_1^2 \frac{q}{\tau} = \frac{1}{\tau} \Sigma_1^2 q = \frac{Q}{\tau},$$

if Q is the total heat received between the states 1 and 2. In the case of a perfect gas the change of entropy in passing from the state 1 to the state 2 may be shown to be

$$(45) \quad \eta_2 - \eta_1 = C_v \log \frac{\tau_2}{\tau_1} + R \log \frac{v_2}{v_1} = C_v \log \frac{p_2}{p_1} + C_p \log \frac{v_1}{v_2},$$

where the symbols have their usual signification.

The natural zero of entropy is that of a body entirely deprived of heat, but as it is not possible to bring bodies into this condition, it is more convenient to reckon entropy from an arbitrary zero, defined by some standard pressure and temperature, just as hights are measured from the arbitrary zero of the sea level.

244. Experimental Determination of Entropy. — Suppose that A (Fig. 186) is the standard state in which the entropy is taken as zero, then the entropy of any other state, B, may be determined as follows: Call τ the temperature of the standard state. Let the body in the state B expand adiabatically until the temperature falls to τ and a state given by the point C. Then, keeping the temperature constant, let the body be brought into the state A. Now, since C is on the same isentropic with B, the entire change of entropy along the path BCA takes place in going from C to A, so that if Q is the heat abstracted in this last change, the entropy at B is given by $\frac{Q}{\tau}$.

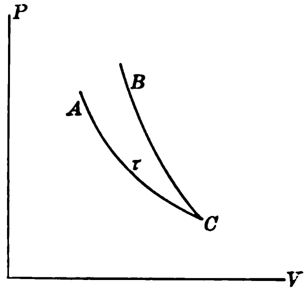


FIG. 186.

245. Change of Entropy by Equalization of Temperature. — If a small quantity of heat, q, pass from a body at the temperature τ_1 into another at a lower temperature, τ_2, the entropy of the first body is diminished by $\frac{q}{\tau_1}$ and that of the second increased by $\frac{q}{\tau_2}$, so that the entropy of the pair is increased by $q\left(\frac{1}{\tau_2}-\frac{1}{\tau_1}\right)=q\frac{\tau_1-\tau_2}{\tau_1\tau_2}$, since $\tau_1>\tau_2$.

Therefore, all transfer of heat by the processes of conduction and radiation from one body of a system to another increases the entropy of the system. Clausius expressed this by saying that "the entropy of the universe tends to a maximum."

TABLE OF HEAT OF COMBINATION WITH OXYGEN.

1 Gram of	Compound Formed.	Calories of Heat Produced.	Equivalent Energy in Ergs.
Hydrogen	H_2O	34000	$1.43(10)^{12}$
Carbon	CO_2	8000	$3.36(10)^{11}$
Sulphur	SO_2	2300	$9.66(10)^{10}$
Phosphorus	P_2O_5	5747	$2.41(10)^{11}$
Zinc	ZnO	1301	$5.46(10)^{10}$
Iron	Fe_3O_4	1576	6.62 "
Tin	SnO_2	1233	5.18 "
Copper	CuO	602	2.53 "
Carbonic Oxide	CO_2	2420	$1.02(10)^{11}$
Marsh Gas	CO_2 and H_2O	13100	5.50 "
Olefiant Gas	" " "	11900	5.00 "
Alcohol	" " "	6900	2.90 "

246. Heat of Combination. — In order to separate two molecules which are chemically united, the expenditure of a definite amount of work is necessary. Accordingly, when

the molecules are allowed again to unite, they do an equivalent amount of work, which usually appears as heat and may be measured by a proper calorimetric apparatus. The heat of combination of oxygen with various substances has been made the subject of elaborate investigation by Andrews and Faure. Their results are shown in the preceding table.

247. Available Energy. — When an engine takes a quantity of heat, Q_S, from a source at the temperature τ_S, and delivers a quantity, Q_R, at the temperature τ_R,

$$Q_S - Q_R = \frac{\tau_S - \tau_R}{\tau_S} Q_S = \left(1 - \frac{\tau_R}{\tau_S}\right) Q_S \tag{46}$$

units are transformed into work, but the remainder Q_R, according to the second law of thermodynamics, is unavailable for the purposes of work, if the refrigerator is the coldest body of the system. If equation 46 be taken as the measure of availability, the dissipation, or diminution of availability of a quantity of heat in passing from a body at the temperature τ_1 to another at τ_2 will be

$$\left(1 - \frac{\tau_R}{\tau_1}\right) q - \left(1 - \frac{\tau_R}{\tau_2}\right) q = \tau_R q \left(\frac{1}{\tau_2} - \frac{1}{\tau_1}\right) = \tau_R (\eta_2 - \eta_1);$$

that is to say, the product of the lowest temperature by the increase of entropy. Therefore, since the entropy of the universe is continually increasing, the available energy is tending toward zero.

248. Sources of Energy Available to Man. — The stores of energy from which we may derive work may be classified as follows:

1°. Fuel.	4°. Wind.
2°. Food.	5°. Tides.
3°. Head of Water.	6°. Solar Radiation.

The source of the energy in each of these cases, except the 5°, may be traced ultimately to the sun. Thus, the different forms of fuel, *e.g.* coal, wood, etc., are products of vegetable growths, which represent for present purposes definite amounts of the compounds of hydrogen and carbon chemically separated from another quantity of oxygen. The exact manner by which, in the physiological processes of plant life, this chemical separation is effected is but imperfectly understood.

In some way chlorophyll, the familiar green coloring matter of plants, appears to be able in the presence of sunlight to decompose carbon dioxide, the carbon being assimilated by the plant and the oxygen set free. The energy which man derives from food is likewise due to chemical separation effected by plant life, and hence traceable to the sun.

The energy which is obtained by allowing water under pressure to flow through a water wheel is derived from the sun by a simple transformation governed by physical laws alone. The water is evaporated and raised by the heat of the sun, and afterward deposited in the form of rain at elevations above the reservoirs in which it is collected.

The kinetic energy of the winds is also derived, almost exclusively, from the heat of the sun. The energy of the tides, sometimes used to turn mill wheels, is due to the kinetic energy of the relative motions of the earth, moon, and sun, combined with the potential energy of mutual gravitation.

EXAMPLES.

1. A mass of mercury falls from a hight of 5 meters. How much would its temperature be raised if all the heat developed were applied to this purpose? *Ans.* 0.35°.

2. A piece of ice weighing 8.25 kil. is dropped into a pool of water at 0°. From what hight must it fall in order to melt 175 gms.? *Ans.* 717 meters.

3. With what velocity must a lead bullet strike a target in order to raise its temperature 50°, on the assumption that one-half the heat generated is lost? *Ans.* 163 meters per sec.

4. With what velocity must a lead bullet at 25° strike a target in order that the former may be melted, assuming that one-half the heat developed is dissipated? *Ans.* 500 meters per sec.

5. 775 cc. of air are heated from 0° to 100° under a constant pressure of one atmosphere. How much work will be done by the expansion? *Ans.* $2.88(10)^8$ ergs.

6. The thermal capacity of a system, consisting of a copper vessel filled with compressed air immersed in a body of water, was 10,680 cal. per degree. When 44,600 cc. of air were allowed to escape at a pressure of 76.5 cm. of mercury, the temperature of the system fell through 0.097°. What value does this give for the mechanical equivalent of heat? *Ans.* $4.39(10)^7$ cm.^{2} / sec.^{2} deg. C.

7. What would be the maximum efficiency of an engine working between 138° C. and 44° C.? *Ans.* 20.4 %.

8. The melting point of sulphur is 115°, and its latent heat 9.3 cal. per gm. The density in the solid state being 2.05 gms. per cc. and 1.95 gms. per cc. in the liquid state, at what temperature will sulphur melt under a pressure of 25 atmospheres? *Ans.* 115.63°.

9. If a specimen of coal contain 88 % carbon and 4.5 % hydrogen, how many calories will be generated by the combustion of 1 gm. of the coal? *Ans.* 8570 cal.

CHAPTER XV.

KINETIC THEORY OF GASES.

249. Explanation of the Pressure of a Gas. — An early suggested explanation of the pressure of a gas was that the molecules exerted a mutual repulsion. It was found, however, that it was impossible to state any law of repulsion which should make the pressure fulfill Boyle's Law, and at the same time be independent of the shape of the containing vessel. Another theory, suggested by Daniel Bernoulli, as early as 1738, was that the pressure of a gas arose from the impact of the molecules against the sides of the containing vessel. This theory was not worked out in definite form till 1848, when Joule showed, from the principles of dynamics, what the velocity of the molecules should be to produce the observed pressure.

250. Expression for the Pressure of a Gas. — The fundamental postulates of the kinetic theory are: 1°, that every gaseous body which can be experimented on contains an indefinitely great number of molecules, all of the same size; 2°, that each of these molecules moves with great velocity in a straight line until it strikes the sides of the vessel, or encounters some other molecule, after which it moves off in a new direction; 3°, that the time of free motion of the molecule is very much greater than that occupied by an encounter; 4°, that the molecules influence each other by impact only. No definite assumption can be made concern-

ing the behavior of the individual molecules, for even if they all had the same velocity initially, this velocity must be altered both in direction and amount at each encounter, so that they must soon appear to move in an entirely irregular manner. However, the application of the laws of chance to such a system of molecules, moving in an entirely fortuitous manner, shows that probably some molecules are moving very slowly, a very few are moving with enormous velocities, and that the others have velocities intermediate between these extremes. The best method of treating these velocities, for the purpose of calculating the pressure, is to take the average of the squares of all the velocities. This quantity is called the *mean square of the velocity*, and will be denoted by $\overline{V^2}$. The square root of this quantity is called the *velocity of mean square*.

In order to find the pressure of a gas, suppose that the latter is contained in a rectangular vessel, and let

a = length of the vessel,
b = breadth of the vessel,
c = depth of the vessel,
p = pressure of the gas,
ρ = density of the gas,
n = number of molecules,
M = mass of each molecule,
t = small time occupied by the impact,
$\overline{V^2}$ = mean of the squares of the velocities of all the molecules,
$U = \sqrt{\overline{V^2}}$ = velocity of mean square.

On the whole, as many molecules may be regarded as moving in one direction as another, *i.e.* one-third may be supposed to be moving parallel to each side of the vessel. Now, if a molecule moving parallel to a, with a velocity U,

be supposed to strike the end of the vessel, and bound back with the same velocity, its momentum will have been changed by an amount

$$MU-(-MU)=2MU;$$

or, since $\frac{n}{3}$ molecules are moving in this direction, the change of momentum consequent upon one impact by all the molecules against the end will be

$$\tfrac{2}{3}nMU.$$

If each molecule be supposed to travel the length of the box without interruption, it will strike the end $\frac{Ut}{2a}$ times in the time t, so that the total change of momentum in the time t is

$$\tfrac{2}{3}nMU\cdot\frac{Ut}{2a}=\tfrac{1}{3}\,\frac{nM\overline{V^2}t}{a},$$

and the force exerted against the end, which is measured by the time rate of change of this quantity, is

$$F=\tfrac{1}{3}\,\frac{nM\overline{V^2}}{a}.$$

Therefore, the pressure on the end whose area is bc will be

$$p=\frac{F}{bc}=\tfrac{1}{3}\,\frac{nM\overline{V^2}}{abc}.$$

But nM is the total mass of the gas and abc is the volume; therefore, since

$$\frac{Mn}{abc}=\rho,$$

(1) $$p=\tfrac{1}{3}\rho\overline{V^2},$$

or, solving for $\overline{V^2}$, the mean square of the velocity is

(2) $$\overline{V^2}=\frac{3p}{\rho}.$$

251. Boyle's Law. — If in equation 1 the volume abc be denoted by v,

$$pv = \tfrac{1}{3} m \overline{V^2}. \tag{3}$$

Now, since $\overline{V^2}$ depends only on the temperature in the same body of gas,

$$pv = \text{constant},$$

for a constant temperature.

Boyle's Law is thus raised from the rank of an experimental fact to a deduction from the simple assumptions made at the beginning of the preceding article.

252. Clausius' Equation of the Virial. — In the case where the influence of one molecule on another is not negligible, Clausius has proved an equation which greatly extends the generality of the investigation concerning the pressure of a gas in the kinetic theory. Let m, $\overline{V^2}$, p, and v have the same meaning as before, but suppose now that each particle exerts an attraction, or repulsion, R, upon another particle at the distance r.

The product Rr is called the *virial* of the stress, and the virial of the system will be the sum of the virial of the stresses which exist in it.

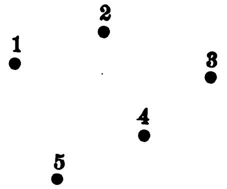


FIG. 187.

Let Fig. 187 represent a number of particles, and let the stress between 1 and 2 be denoted by R_{12}, and the distance by r_{12}, then the virial for all the particles taken with respect to 1 will be

$$R_{12}r_{12} + R_{13}r_{13} + \cdots + R_{1n}r_{1n} = \Sigma_1^n R_{1k}r_{1k},$$

where the index k is to be given all values from 1 to n, and likewise for the particle 2

$$R_{21}r_{21} + R_{23}r_{23} + \cdots + R_{2n}r_{2n} = \Sigma_1^n R_{2k}r_{2k},$$

and so on. Adding these expressions, the virial for the whole system becomes

$$\tfrac{1}{2}\Sigma_1^n\Sigma_1^n R_{kk} r_{kk},$$

the $\frac{1}{2}$ being taken because $R_{12} = R_{21}$, etc.

Clausius' equation for the virial may be written

$$(4) \qquad \tfrac{1}{2}\Sigma M\overline{V}^2 = \tfrac{3}{2}pv + \tfrac{1}{2}\Sigma\Sigma Rr,$$

in which the left-hand member denotes the kinetic energy of the particles, and the right-hand side the value found for this energy in equation 3 increased by the virial of the system. In gases like hydrogen, air, etc., the virial is very small under ordinary conditions, but if the gas be compressed it may naturally be expected to change.

Solving (4),

$$(5) \qquad pv = \tfrac{1}{3}\Sigma M\overline{V}^2 - \tfrac{1}{3}\Sigma\Sigma Rr,$$

and, comparing with the results of Art. 182, it appears that the virial is at first positive, but changes sign as p increases; that is to say, the force between the particles is first an attraction, but afterward becomes a repulsion.

253. Van der Waals' Equation. — Van der Waals has shown how to take account of the size of the molecules, and also of those forces which give rise to the phenomena of surface tension in a liquid. Boyle's Law, $pv = mR\tau$, would indicate that if the pressure were sufficiently increased the volume would approach zero, a conclusion which is not consistent with the known properties of matter.

If the minimum value of the volume be called b, the expression for the pressure,

$$\frac{mR\tau}{v-b},$$

will be free from the objection just mentioned.

Also, if it be assumed that different portions of the fluid attract one another, the simplest supposition that can be made will be that this force is proportional to the mass of the elementary portions into which the body may be supposed to be divided. But the quantity of matter in an element will vary as the density $\rho = \frac{m}{v}$, where m is the mass of the body and v its volume, so that the attraction F of the force which gives rise to the surface tension may be written

$$F \propto \frac{m}{v} \cdot \frac{m}{v}.$$

Subtracting this term from the previous value for the pressure, since it diminishes it, Van der Waals wrote

$$p = \frac{mR\tau}{v-b} - a\frac{m^2}{v^2}, \text{ or}$$

$$\left(p + \frac{am^2}{v^2}\right)(v-b) = mR\tau, \tag{6}$$

where a is some constant. This equation very well represents the experimental results of Andrews, presented in Art. 180.

254. Calculation of Molecular Velocities.— The equation

$$\overline{V^2} = \frac{3p}{\rho}$$

permits the velocity of mean square to be calculated at once. Thus, for air at 0° C.,

$$p = 1.013(10)^6 \frac{\text{dynes}}{\text{cm.}^2},$$

$$\rho = \frac{1}{773} \frac{\text{gm.}}{\text{cm.}^3},$$

whence, $$\sqrt{\overline{V^2}} = 4.85(10)^4 \frac{\text{cm.}}{\text{sec.}}.$$

Observing that for different substances this velocity varies inversely as the density, its value for any other gas may evidently be obtained by multiplying the value just found by the square root of ratio of the density of air to that of the gas required. Thus, for hydrogen, which is 14.44 times as light as air, the velocity of mean square will be

$$4.85(10)^4 \times \sqrt{14.44} = 1.84(10)^5 \, \frac{\text{cm.}}{\text{sec.}},$$

or nearly a mile per second.

The molecular velocity of a gas bears an important relation to the elements which constitute the matter of the earth. If it were possible to give a projectile directed away from the earth a sufficiently great velocity, it would soon pass into a region where the earth's attraction would be too small to draw the body back to its surface. Now, as this velocity may be shown to be not much greater than that of the hydrogen molecule, it seems probable that if the earth ever possessed elements less dense than hydrogen they have now escaped into space.

At the surface of the moon weight has but one-fifth the value of terrestrial weight, since the diameter of the moon is only one-fourth and its mass one-eightieth of that of the earth. Consequently, the lightest substance which the moon can retain must have a density twenty-five times that of hydrogen. Now, as most substances which are likely to exist in the aeriform condition are lighter than this, it might be predicted that the moon would have little or no atmosphere. This conclusion is verified by observation, which shows that, if the moon possesses an atmosphere, it cannot exert a pressure exceeding 1 mm. of mercury.

On the other hand, it is possible that a body as large as the sun may possess elements much lighter than hydrogen.

255. Law of Avogadro. — If two sets of molecules of different masses are moving in a vessel, there will result from their successive encounters an exchange of energy up to a certain point. It has been shown by Maxwell that equilibrium between two such systems will occur when the mean kinetic energy of a single molecule of each set is the same. Thus, if the molecular masses are M_1, M_2, and the mean squares of their velocities are, respectively, $\overline{V_1^2}$ and $\overline{V_2^2}$, this condition of equilibrium may be stated

$$\tfrac{1}{2} M_1 \overline{V_1^2} = \tfrac{1}{2} M_2 \overline{V_2^2}. \tag{7}$$

If the pressure of the gases be denoted by p_1, p_2, the number of molecules by n_1, n_2, and the volumes by v_1, v_2,

$$p_1 = \tfrac{1}{3} \frac{n_1 M_1 \overline{V_1^2}}{v_1},$$

and

$$p_2 = \tfrac{1}{3} \frac{n_2 M_2 \overline{V_2^2}}{v_2}. \tag{8}$$

If the volumes and pressures are the same for both gases,

$$n_1 M_1 \overline{V_1^2} = n_2 M_2 \overline{V_2^2}. \tag{9}$$

If, in addition, the gases are in thermal equilibrium, *i.e.* have the same temperature, by equation 7

$$M_1 \overline{V_1^2} = M_2 \overline{V_2^2},$$

whence,

$$n_1 = n_2. \tag{10}$$

That is to say, in equal volumes of any two gases at the same temperature and pressure there is an equal number of molecules. This law was announced by Avogadro in 1811.

If the density of the gases at the same temperature and pressure be denoted by ρ_1, ρ_2,

$$\rho_1 = \frac{n_1 M_1}{v_1}, \quad \rho_2 = \frac{n_2 M_2}{v_2}, \tag{11}$$

by definition. Then, since by equation 10 the volumes are proportional to the number of molecules, it follows that

$$\frac{\rho_1}{\rho_2} = \frac{M_1}{M_2}, \tag{12}$$

or the masses of the molecules are proportional to the absolute densities of the gases measured at the same temperature and pressure.

Since, for thermal equilibrium, by definition, two bodies have the same temperature, equation 7 will be satisfied if the absolute temperature is taken proportional to $M\overline{V}_2^2$. The same result is obtained by comparing

$$pv = \tfrac{1}{3} nM\overline{V}^2$$

with

$$pv = mR\tau.$$

256. Gas Density and Molecular Masses. — The ratio of the density of a gas to the density of hydrogen is called its *gas density*.

Let D = gas density of the gas,
ρ = absolute density of the gas,
ρ_H = " " " hydrogen,
M = mass of a molecule of the gas,
M_H = " " " " " hydrogen.

Then, by the definition and equation 12,

$$\begin{cases} D = \dfrac{\rho}{\rho_H} = \dfrac{M}{M_H}, \text{ or,} \\ M = DM_H. \end{cases} \tag{13}$$

That is, the molecular mass of a gas can be found in terms of the hydrogen molecule when the gas density is known. Molecular masses are, however, usually expressed in terms of the hydrogen atom, *i.e.* half of the hydrogen molecule, which is shown to be divisible into two parts by exposing a mixture of equal volumes of hydrogen and chlorine to full sunshine. A new substance, known as hydrochloric acid, is then formed without change in pressure; whence it follows, by the law of Avogadro, that the number of molecules in the new gas is the same as in the mixture. But since after the reaction the molecule of the acid is certainly compound, for two volumes of hydrochloric acid may be decomposed into one volume of hydrogen and one volume of chlorine, it follows that the molecule in the simple gases must have consisted of two parts, and the reaction was due to the union of each of the parts of the hydrogen molecule uniting with one of the parts of the chlorine molecule. Accordingly, if the mass of the hydrogen atom be denoted by $[h]$,

$$M_H = 2\,[h],$$

and

$$M = 2D\,[h],$$

which is the familiar rule of the chemist that *the molecular mass of any substance is twice its gas density.*

257. Determination of Gas Density. — The gas density of substances which exist ordinarily in the solid or liquid state, and which may be made to pass into the aeriform condition at a temperature below that at which glass softens, may be conveniently determined by a method devised by Victor Meyer. The apparatus consists of a cylindrical glass vessel, B (Fig. 188), drawn out into a long stem, E, which ends in an enlarged mouthpiece, F, closed with a rubber stopper. A little below this end a branch tube, A, with upturned end,

is arranged so as to dip below the surface of a vessel of water. Surrounding the bulb B is a larger vessel, C, of glass, or sometimes of iron, which is intended to contain a hot liquid bath. To determine the gas density of a substance, an amount which when vaporized will not more than half fill the vessel B is weighed out and placed in a flask small enough to pass through the stem E. The bulb is now heated by immersion in a bath at a suitable temperature. During this operation some of the air is driven out and escapes through the branch tube. When the whole has reached a constant temperature, and there is no further escape of air from A, the stopper is removed from F and the flask containing the substance is dropped into B, at the bottom of which a cushion of asbestos has been placed to prevent fracture of the glass. The stopper is at once replaced, and after one or two bubbles of air, forced out by the introduction of the stopper, have escaped from A, a graduated tube, D, filled with water, is carefully inverted over the orifice. The substance in the flask will be vaporized in a few seconds and drive out its own bulk of air which is collected in D. As soon as equilibrium is attained, the graduated tube is lowered until the level of the water is the same within and without, *i.e.* the pressure of the air in the tube is the same as that of the external atmosphere. The hight of the barometer is then noted, together with the temperature of the air collected and its volume, which may be read from the graduations of the tube.

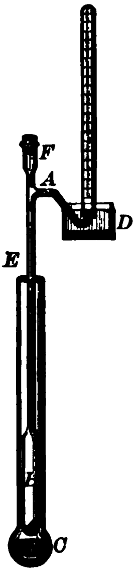


FIG. 188.

Let v = volume of the air collected,

m' = mass of the air collected,

$t =$ temperature of the air collected,
$h =$ pressure of the air collected in cm. of mercury,
$w =$ pressure of water vapor at $t°$ C. in cm. of mercury,
$m =$ mass of substance vaporized,
$D =$ gas density of substance vaporized,
$D' =$ gas density of air $= 14.44$.

The volume v may be reduced to zero and 76 cm. pressure, by the equation 23, Chapter IX; thus,

$$v_o = \frac{h - w}{76 \text{ cm.}} \frac{v}{1 + \frac{t}{273}}, \tag{14}$$

h being corrected for the pressure of the water vapor within the tube.

Then, since the density of air is $\frac{1}{773} \frac{\text{gm.}}{\text{cc.}}$, the mass of the air will be

$$m' = \frac{v_o}{773} \frac{\text{gm.}}{\text{cc.}}; \tag{15}$$

and, finally, since the masses of equal volumes of gases are as their densities,

$$\begin{cases} \dfrac{D}{D'} = \dfrac{m}{m'}, \text{ or,} \\ D = \dfrac{m}{m'} 14.44. \end{cases} \tag{16}$$

258. Diffusion. — If a small hole of area s be made in the end of the box discussed in Art. 250, the rate of escape may be easily calculated as follows: One molecule strikes the end $\frac{V}{2a}$ times a second, so that if $\frac{1}{3}$ of the molecules be supposed to move parallel to a, the end whose area is bc will

sustain $n\frac{V}{6a}$ blows per second; that is to say, the rate of escape from the area s will be

$$\frac{nVs}{6abc}.$$

Since n is by Avogadro's Law the same for all gases under the same conditions, the rate of escape will be proportional to the velocity of mean square, or inversely as the square root of the density of the gas. This conclusion is found to be verified by experiment, and thus offers the strongest possible proof of the kinetic theory.

The plaster plate used in the experiment on diffusion, described in Art. 191, is to be regarded simply as a wall with innumerable holes. The excess of pressure on one side of the plate is obviously a differential effect depending on the fact that more of the lighter molecules pass in one direction than of the denser ones going the opposite way, for the mixtures in time become alike on both sides and the difference of pressure vanishes. If, however, a gas, a, could be placed in a vessel, A (Fig. 189), made of such material that its walls should be absolutely impervious to it, and then were surrounded by another vessel, B, containing a gas, b, to which the walls of A were completely permeable, it is evident that the gas b would diffuse into A until it exerted the same pressure on both sides of the wall. The total pressure in A would, however, exceed that in B by exactly the pressure of the gas a. Although such an experiment is not realizable in the case of gases, it is in the case of liquids (see Art. 192), the excess of pressure being then known as osmotic pressure, the

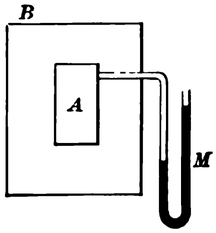


FIG. 189.

explanation and the numerical value of which, as has been experimentally shown, are precisely the same as for gaseous pressure.

259. Energy of the Gaseous Molecule. — The energy of a molecule will in general consist of three parts: 1°, that due to the velocity of translation, already designated by $\frac{1}{2}M\overline{V}^2$; 2°, that due to rotation among the component parts of the molecule; and 3°, that due to the vibration of these parts.

If it be assumed that each of these portions is proportional to the energy of translation, the total energy may be written

$$\tfrac{1}{2}\beta M\overline{V}^2,$$

where β is a constant greater than unity and probably equal to 1.634 for air, and the more perfect gases.

The total kinetic energy of the substance will be found by multiplying by the number of molecules; thus,

$$T = \tfrac{1}{2}\beta n M\overline{V}^2, \tag{17}$$

comparing with

$$p = \tfrac{1}{3}\frac{nM\overline{V}^2}{v},$$

$$T = \tfrac{3}{2}pv\beta, \tag{18}$$

or, the kinetic energy per unit volume is

$$T_v = \tfrac{3}{2}\beta p. \tag{19}$$

260. Law of Dulong and Petit. — The specific heat at constant volume C_v is numerically the quantity of heat required to raise 1 gm. through 1° C., when the volume does not change. Now, since pv is proportional to the absolute temperature, the total kinetic energy of a body of gas will be proportional to the absolute temperature. Hence, the

increase of energy of the unit mass per degree rise of temperature will be found by dividing T by m and τ; thus,

$$\tfrac{3}{2}\frac{pv\beta}{m\tau},$$

which is reduced to heat units by dividing by J. Therefore,

$$C_v = \tfrac{3}{2}\frac{pv\beta}{Jm\tau} = \tfrac{3}{2}\frac{pv\beta}{nMJ\tau}, \text{ or,} \tag{20}$$

$$MC_v = \tfrac{3}{2}\frac{R\beta}{nJ} = \text{constant}, \tag{21}$$

if β is constant. This is the law of Dulong and Petit.

261. Determination of β. — By equation 23, Art. 240,

$$J(C_p - C_v) = R,$$

or

$$\frac{C_p}{C_v} - 1 = \frac{R}{C_v J}.$$

Substituting the values of R and C_v, and calling $\frac{C_p}{C_v} = \gamma$,

$$\left\{\begin{aligned} \gamma - 1 &= \tfrac{2}{3}\beta, \text{ or,} \\ \beta &= \tfrac{2}{3}\frac{1}{\gamma - 1}. \end{aligned}\right. \tag{22}$$

The value of γ is usually found from the velocity of sound in the gas by the equation

$$u^2 = \gamma\frac{p}{\rho} \tag{23}$$

(see Art. 494), where u is the velocity of sound and ρ is the density of the gas.

Combining equation 23 and $\overline{V^2} = \frac{3p}{\rho}$, the velocity of mean square is found to be

$$(24) \qquad \sqrt{\overline{V^2}} = u\sqrt{\frac{\gamma}{3}},$$

or
$$\sqrt{\overline{V^2}} = 1.458\, u \text{ in gases, for which}$$
$$\gamma = 1.408.$$

262. Viscosity of Gases. — The viscosity of a gas is a physical property closely connected with diffusion. Suppose that a stratum of a fluid is moving above the fixed horizontal plane AB, in the direction of the arrow (Fig. 190), with a velocity proportional to the distance from the plane, and let the velocity at the distance d be called V. The substance between the planes is undergoing a shear which increases at the rate $\frac{V}{d}$.

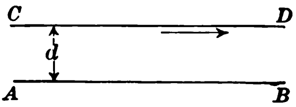


FIG. 190.

If the shearing stress is denoted by S, the coefficient of viscosity may be defined by

$$(25) \qquad \mu = \frac{S}{\frac{V}{d}} = \frac{Sd}{V}.$$

Its dimensions are $\frac{M}{LT}$.

A coefficient of viscosity per unit density is also sometimes used. Let it be denoted by ν; then

$$(26) \qquad \nu = \frac{\mu}{\rho},$$

and its dimensions are $\frac{L^2}{T}$.

This last coefficient is connected with K, the coefficient of diffusion of a gas into itself, by the equation

$$\nu = 0.6479\,K. \tag{27}$$

The viscosity of a gas determines the rate at which fine particles will settle through it. Calculations by Stokes indicate that a drop of water $\frac{1}{1000}$ of an inch in diameter will fall through air, a thousand times rarer than itself, with a velocity of only $\frac{8}{10}$ of an inch per second; but if the drop were $\frac{1}{10}$ of this size its velocity would be a hundred times smaller, or, approximately, $\frac{1}{2}$ an inch per minute. When the mean density of a cloud of such particles is less than that of air containing water vapor, the cloud will ascend. In general, the motion of fine particles through a fluid is a very slow process, and varies inversely as the square of the diameter of the particles.

263. Size and Number of Molecules. — Since it is found to be impossible to separate any simple gas by diffusion through a porous septum into portions having different densities, the important conclusion may be drawn that the molecules of such a gas are all of like mass.

The average distance which a molecule goes before encountering another molecule is called its *mean free path*. It evidently depends on the average space occupied by the molecule and its cross section, for the chance of one molecule striking another increases in proportion as the molecules are brought nearer together by increasing the density, and is greater with molecules of greater cross section.

Let v = volume of the gas,
V_M = " " a molecule,
D_M = diameter of a molecule,
S = mean free path.

Boltzmann has shown that these four quantities are connected by the equation

$$\frac{\frac{1}{8}D_M}{S} = \frac{V_M}{v}. \tag{28}$$

Now, although there is no way of determining the cross section of the molecules exactly, the great incompressibility of liquids seems to indicate that the molecules are not far removed from contact. A probable estimate of V_M may, accordingly, be obtained from the volume occupied by a given mass of gas when condensed into the liquid form.

The value of S may be determined with considerable precision from the viscosity of the gas, so that the approximate diameter of the molecule may be calculated from equation 28. The following table from Maxwell shows some of the important molecular constants.

	HYDROGEN.	OXYGEN.	CO.	CO_2.
Velocity of mean square in $\frac{\text{cm.}}{\text{sec.}}$	$1.859\,(10)^5$	$0.465\,(10)^5$	$0.497\,(10)^5$	$0.396\,(10)^5$
Mean free path in cm. . . .	$9.65\ (10)^{-6}$	$5.60\ (10)^{-6}$	$4.82\ (10)^{-6}$	$3.79\ (10)^{-6}$
Diameter of Molecule in cm. . . .	$5.8\ (10)^{-8}$	$7.6\ (10)^{-18}$	$8.3\ (10)^{-8}$	$9.3\ (10)^{-8}$
Mass of Molecule in grams . .	$4.6\ (10)^{-24}$	$7.36\ (10)^{-23}$	$6.44\ (10)^{-23}$	$1.012\,(10)^{-22}$

As these values for the size of the molecules do not differ much from estimates obtained from entirely distinct methods (see Art. 112), they are entitled to considerable confidence. It seems hardly possible that they are ten times too great or ten times too small.

Knowing the mass of a molecule, the number of molecules in a cubic centimeter of the gas may be found at once from the density.

Thus, $$N = 21(10)^{18}$$

for all gases at a pressure of one atmosphere and a temperature 0° C. Since there is no means of reducing the pressure of a gas much more than 100,000 times, the rarest substance which can be experimented on will still contain something like $21(10)^{13}$ molecules per cubic centimeter, that is, practically an infinite number.

PART III. — ELECTRICITY.

CHAPTER XVI.

ELECTRIFICATION.

264. Phenomena of Electrification. — When two dissimilar substances, *e.g.* glass and silk, are rubbed together, they acquire the power of attracting light bodies, such as bits of paper or pith. This phenomenon is not explicable in terms of any other known property of matter, but may be ascribed to a distinct physical entity called *electricity*. The name is derived from *ἤλεκτρον*, the Greek word for amber, a fossil resin in which electrical phenomena were first observed.

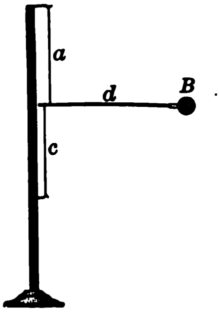


Fig. 191.

Bodies so excited as to exhibit electrical forces are said to be *electrified*, or charged with electricity. Their properties may be conveniently studied by the aid of a horizontal pendulum shown in Fig. 191. B is a very light gilded ball, fastened to the end of a glass rod, d, and so suspended by threads a and c that it will, under the influence of weight, perform a slow horizontal oscillation about its position of rest. If a glass rod, excited by rubbing with a piece of silk, be presented to the ball, the latter will be attracted till it touches the rod, after which it exhibits an evident repulsion from the rod, indicating that the ball has itself acquired electrical properties from the contact. If,

now, a piece of vulcanite or resinous substance be rubbed with cat's fur, and approached to the ball, the latter will be strongly attracted by the vulcanite, though still repelled by the electrified glass. If the uncharged ball be first touched with the electrified vulcanite, it will be repelled by the vulcanite but attracted by the glass.

These experiments establish the following fundamental principles: 1°, that there are two kinds of electrification; 2°, that like charges repel; and 3°, that unlike charges attract.

The charge which is found on the glass in the treatment specified may, for distinction, be called *vitreous*, or *positive;* and that on the vulcanite, *resinous*, or *negative.*

Electrification may be produced in a variety of ways other than by rubbing, as, *e.g.* by percussion, by cleavage, by pressure, by heating, and by change of state; but the characteristics of the electrification, however produced, differ in no respect from those just described.

265. Conduction. — If the charged ball in Fig. 191 be touched with the finger or with a wire, all evidence of electrification will disappear. During the period of discharge the wire is regarded as conveying a quantity of electricity. If, however, the ball be touched with a piece of unelectrified glass, the charge of the ball will remain sensibly unaltered.

Bodies may be conveniently divided into three classes, according to their ability to conduct electricity:

1°. Conductors, or bodies which readily permit the passage of electricity. They are characterized by their opacity to light. Metals are the best conductors, but no substance can be regarded as a perfect conductor.

2°. Electrolytes, or liquids which are chemically separated into two components by the passage of electricity. Aqueous

solutions of the metallic salts are examples of electrolytes. The transfer of electricity by such bodies is called electrolytic conduction, and bears a close analogy to the process of convection in heat.

3°. Non-conductors, or bodies which will retain an electric charge for a long time. They are characterized by their translucency, at least in thin plates. Gases, gums, and glass-like bodies are examples of non-conductors or insulators. No body is a perfect insulator, nor is there any line of division between the above-named classes.

266. Gold-Leaf Electroscope. — A form of electroscope more sensitive than that described in Art. 264 is shown in Fig. 192. It consists of a brass cylinder, *A*, mounted on a suitable stand and closed at the ends by glass plates. A brass stem, with projecting knob, passes through the upper side of the cylinder, but is carefully insulated from it by a ring of shellac. To the bottom of this stem are fastened two strips of gold leaf, which diverge when any charged body is brought into the vicinity of the electroscope. If a charge be first communicated to the gold leaves, they will diverge still further on the approach of a body similarly charged, but will retract if the body is oppositely electrified.

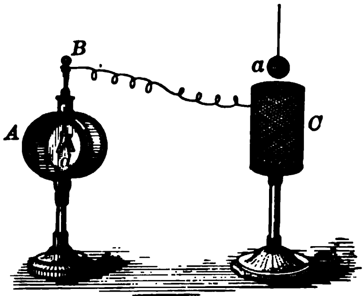


FIG. 192.

267. Faraday's Ice-Pail Experiment. — Let *C* (Fig. 192) represent a metallic vessel, supported on an insulating stand and connected by a wire with the electroscope *B*. If, now, a metal ball, *a*, suspended

by a silk thread and charged with electricity, be lowered into the vessel, the following observations may be made:

(*a*) That the outside of the vessel is electrified, and has a charge of the same sign as that on the ball.

(*b*) That the position of the ball within the vessel, provided it is well below the mouth, does not affect the gold leaves.

(*c*) That on removal of the ball, without contact with the sides, all sign of electrification disappears.

(*d*) That contact of the ball against the inner surface of the vessel produces no change in the divergence of the gold leaves.

(*e*) That after subsequent contact with the inner surface of the vessel, or on removal, the ball is entirely without charge.

These experiments, first performed by Faraday with a pewter ice-pail, lead to the following conclusion of fundamental importance.

Observation (*a*) shows that an electrified body may produce electrification in the vessel while separated from it by a non-conducting medium. Such a medium, considered as transmitting these electrical effects without conduction, Faraday called a *dielectric*, and the action which takes place through it, *induction*. Observation (*c*) shows that the vessel had not then received a permanent charge. If, however, the vessel had been touched to earth, while under the inductive action of the ball, the leaves of the electroscope would have at once collapsed. But on removal of the ball, after breaking the earth connection, the leaves would be found to diverge with a charge opposite to that on the ball. This process is known as charging by induction.

Observations (*d*) and (*e*) together show that the electrification of the ball was exactly equal and opposite to that on

the inner surface of the vessel, for one just neutralized the other.

Observation (*e*) shows that there is no charge on the inner surface of the vessel.

268. Quantity of Electricity. — The method of charging by induction, just explained, shows how it is possible to obtain a charge exactly equal and opposite to that of an electrified body, without altering the electrification of the latter. It is thus possible to electrify any number of metallic vessels with equal quantities which may be taken arbitrarily as units. By introducing a number of such charged vessels within a larger insulated vessel, an amount of electricity which is any multiple of this unit may be induced on the latter.

269. Conservation of Electricity. — Electricity in all phenomena behaves like an incompressible fluid; that is to say, if an amount of electricity appears in one place, an exactly equal amount disappears from another place. In other words, it can neither be created nor destroyed. This proposition may be illustrated by the following experiment.

Let a rod of glass and a rod of vulcanite, which have been completely discharged by passing them several times rapidly through a flame, be rubbed together within the metallic vessel of Fig. 192, care being taken not to touch them to the vessel. On removing the glass rod the vulcanite rod will be found to produce a divergence of the leaves of the electroscope, and in the same manner, if the vulcanite is removed while the glass is held within the vessel, the leaves will again diverge, but with the opposite sign, as may be readily shown by approaching any known charge to the gold leaves.

But it will always be found that, as long as both rods are within the cage, there is no sign of electrification on the gold leaves.

This experiment shows that, since the charges of the glass and the vulcanite must have been equal and opposite in order to produce entire neutrality of the leaves, electrification may be regarded in the nature of a separation, electricity having passed from one rod to the other. It is not known which body in this experiment has received the excess of electricity, but it is arbitrarily assumed that it is the glass. For this reason the vitreous charge is called positive and the resinous negative.

It is further evident that no electricity was created by the rubbing, for the electroscope in this event must have been affected. Neither was electricity in the rods destroyed by passing them through the flame, for on bringing them again into contact within the vessel they became electrified, but not from any external source. Thus, electricity may be said to be conserved in the same sense that matter and energy are.

270. Faraday's Hollow Cube. — The indications of the ice-pail experiment, that there is no electrical force within a charged conductor, was tested by Faraday in another experiment on a considerably larger scale. He built a hollow cube measuring 12 feet each way, and having covered it with tin foil, insulated it from the earth and charged it to a high degree. Then going into the box and testing its electrical state with the most delicate electroscopes, he could not discover the slightest trace of electrification, although externally there might be at the same time the most striking evidences of violent changes of electrical condition.

271. Law of Electrical Force. — The force between two small electrified bodies varies —

1°, directly as the charge on each body;

2°, with the nature of the dielectric;

3°, inversely as the square of the distance between them.

The first proposition may be verified by arranging two small bodies so that the force between them may be measured by any convenient dynamometer, and giving one of them a fixed charge, q', and the other an amount, q, measured in the provisional units of the article 268. On doubling the quantity q, the force will be found to be doubled, and so on. Since either body might contain the fixed charge, the force may be written

$$F \propto qq',$$

provided that nothing but the charges is varied.

The second proposition may be established by interposing a plate of sulphur, or other insulating material, between the two charged bodies. The force between them will then be found to be less than for the dielectric air. The dependence of the force upon the medium is usually written

$$F = \propto \frac{1}{K},$$

where K is the dielectric constant, regarded as unity for air but greater for most other substances.

The law of the inverse square of the distance may be derived from the fact that there is no resultant electrical force within a charged conductor, as proved by Faraday's experiment with the hollow cube, and that the electricity resides entirely at the surface of a conductor. This last-named fact is shown by the exact equivalence in all electrostatic relations of a solid conductor and a non-conductor covered with a conducting film as thin as $\frac{1}{80000}$ of a centimeter, each of the bodies having the same external dimensions. Thus, let Fig. 193 represent a spherical conductor having a positive

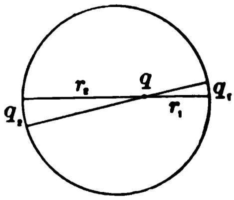


FIG. 193.

charge, Q, upon its surface, and call the surface density, *i.e.* the quantity of electricity per unit area, σ.

The distribution will evidently be uniform, because from the symmetry of the figure there is no reason why a unit of area in one part should have more electricity than in another. Suppose that a small charge, q, is placed at some point within the sphere, and let small cones be drawn having q for their common vertex, and intercepting areas A_1 and A_2 on the surface of the sphere. Let the quantity of electricity on these areas be called q_1 and q_2, and their distances from q, respectively, r_1 and r_2. Also, let F_1 denote the force of repulsion between q and q_1, and, similarly, F_2 that between q and q_2. By approaching a charged body to any electroscope, such, for instance, as that in Fig. 191, it will appear that the repulsion is some inverse function of the distance and greater than the first power of r. Let this function be denoted by $f(r)$; then, by what precedes,

(1) $$F_1 = \frac{qq_1}{Kf(r_1)},$$

and

(2) $$F_2 = -\frac{qq_2}{Kf(r_2)}.$$

But by Art. 270 the resultant force within a conductor vanishes, *i.e.*

(3) $$F_1 + F_2 = 0,$$

whence

(4) $$\frac{qq_1}{Kf(r_1)} - \frac{qq_2}{Kf(r_2)} = 0,$$

or

(5) $$\frac{q_1}{q_2} = \frac{f(r_1)}{f(r_2)}.$$

Also by definition,

$$(6)\qquad \begin{cases} q_1 = \sigma A_1, \\ q_2 = \sigma A_2, \end{cases}$$

therefore,

$$(7)\qquad \frac{q_1}{q_2} = \frac{A_1}{A_2} = \frac{r_1^2}{r_2^2},$$

since areas of similar figures are squares of their homologous lines. Comparing (5) and (7),

$$(8)\qquad \frac{f(r_1)}{f(r_2)} = \frac{r_1^2}{r_2^2};$$

that is to say, the function of r is the square.

This proof was worked out by Cavendish, before 1775, though not published. By a careful investigation of the magnitude of the error committed in his experiments, Cavendish further showed that the assumption that the force vanishes within a charged conductor could not be so far wrong as to make the exponent of r differ from 2 by ± 0.02.

The law of the inverse square was first published in 1787 by Coulomb, who had verified it very roughly by measuring the electrical forces at different distances. Modern investigation has shown that the error in the exponent 2 cannot exceed $\frac{1}{21600}$.

Collecting the preceding results, the law of the mutual action of one charge on another may be written,

$$(9)\qquad F = \frac{qq'}{Kr^2},$$

where F is the force reckoned positive for repulsion, q and q' are the charges, r is the distance between them, and K the dielectric constant.

272. Unit Quantity of Electricity. — If two indefinitely small bodies having equal charges repel each other with the

force of one dyne when placed at the distance of one centimeter in air, the charge on each body is defined as the unit quantity of electricity. The discussion of the dimensions of this and other electrical magnitudes will be found in Chapter XXV.

EXAMPLES.

1. Two small spheres separated by a distance of 14.9 cm. in air are given charges of 39.8 and 45.1 C. G. S. units, respectively. What is the force of repulsion between them? *Ans.* 8.09 dynes.

2. Two small electrified bodies separated a distance of 12.3 cm. in a medium whose dielectric constant is 2.7 are found to attract each other with a force of 7.6 dynes. The charge on one of the bodies is 64 units. What is the charge on the other?
Ans. 48.5 $\text{gm.}^{\frac{1}{2}}\ \text{cm.}^{\frac{3}{2}}$ / sec.

3. What is the distance between two small bodies charged respectively with 75 and 68 units, which repel one another with a force of 8.4 dynes in air? *Ans.* 24.6 cm.

4. Two small pith balls, each weighing 0.025 gm., are suspended from the same point by silk fibers 80 cm. long. When the balls are equally electrified, they separate under the mutual repulsion to a distance of 10.2 cm. What is the charge on each ball?
Ans. 12.7 $\text{gm.}^{\frac{1}{2}}\ \text{cm.}^{\frac{3}{2}}$ / sec.

5. Two small electrified spheres, A, C, having charges respectively of 64 and 144 C. G. S. units, are placed 24 cm. apart. Where in the line AC can a third charge, B, be placed, so that it will be in equilibrium?

$$\textit{Ans.}\ \begin{cases} AB = 9.6 \text{ cm., or } 48 \text{ cm.} \\ CB = 14.4 \text{ cm., or } 72 \text{ cm.} \end{cases}$$

CHAPTER XVII.

THE ELECTRIC FIELD.

273. Electric Field. — Any region through which electrical forces are regarded as acting is called an electric field. The space may be occupied by air or other bodies, or it may be entirely devoid of ordinary matter.

274. Line of Force. — A line drawn through an electric field, so as to have everywhere the direction in which a small positive charge of electricity would be urged, is called a line of electrical force. It follows from the definition that the lines of force will be directed away from a positively charged body and toward a negatively charged one; and, further, since a positive charge can never be developed without the simultaneous appearance of a negative charge somewhere else, the lines of force must terminate either on the bodies in the neighborhood, or on the walls of the room, or on more remote bodies; and where they terminate there is a quantity of electricity exactly equal and opposite to that on the part of the body from which they proceeded.

275. Intensity at a Point. — The intensity at a point, or the strength of the field, is the force exerted on a small charged body divided by the number of units of electricity which it contains. It is numerically the force which would be experienced by a unit charge placed at the point in question. The intensity at a distance, r, from a point charged with a quantity, q, would be

$$\mathcal{F} = \frac{q}{Kr^2}. \tag{1}$$

276. Potential at a Point. — The potential at a point is defined as the work done per unit quantity of electricity in bringing a positive charge from infinity to the point considered. If W be the work done in bringing a body containing the charge q from infinity, or other place where the force vanishes, and V be the potential, then

$$V = \frac{W}{q}. \tag{2}$$

Potential plays much the same rôle in electricity that temperature does in the theory of heat, or that pressure does in treating of fluids. For as heat flows from places of high to low temperature, or a fluid from high to low pressure, so electricity will flow along a conductor from places of high to low potential.

An equipotential surface is a surface every point of which has the same potential. The surface of a conductor is always an equipotential surface if the electricity is at rest, for, by definition, a conductor will not maintain a difference of potential between two points.

277. Representation of an Electrical Field. — By drawing lines of force and equipotential surfaces throughout a field, according to an accepted convention, it is possible to represent with mathematical exactness all the electrical quantities concerned. It is usual to draw the equipotential surfaces for a constant difference of potential, and the lines of force so that if the charged surface be divided into elements having unit area, the number of lines of force which start from each element will be proportional to the number of units of charge upon it.

According to the views of Faraday, the explanation of the mechanical action of one body upon another is to be sought,

not in forces of attraction and repulsion acting at a distance, but in stresses in the medium, of the nature of tension along the lines of force and compression at right angles to them.

Let A (Fig. 194) be a positively charged point, and suppose that the negative charge corresponding to it has been removed to an infinite distance.

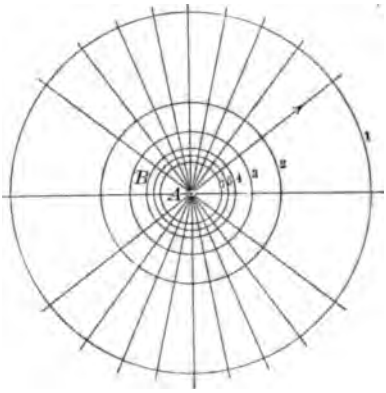


FIG. 194.

It will be observed in this case that the lines of force radiate from the point. The scheme of drawing them in the figure is such that, if the surface of the sphere of unit radius about the point be divided first into zones by a system of planes perpendicular to the horizontal axis, and then into elements of unit area by a system of planes passing through this axis, the number of lines standing on each element is proportional to $\frac{q}{4\pi}$, where q is the charge at the point.

Also, since the work done in bringing up a unit charge from infinity to a point at a distance, r, from A will be the same from whichever side the body is approached, the equipotential surfaces will be a system of concentric spheres, necessarily nearer together as r is diminished, as will be definitely proved in Art. 282.

The intensity at any point of an electric field is inversely proportional to the distance between successive equipotential surfaces. This may be shown as follows: Let d be the distance between surfaces having, respectively, potentials

V_1 and V_2, and $\mathcal{F}$ the mean value of the intensity along the line d; then, by definition,

$$V_1 - V_2 = \mathcal{F} \cdot d,$$

whence,

$$\mathcal{F} = \frac{V_1 - V_2}{d}; \tag{3}$$

that is to say, the intensity of an electric field at a point is equal to the (space) rate of change of the potential at that point.

Equipotential surfaces and lines of force always intersect at right angles; for, suppose that AB (Fig. 195) is a portion of any equipotential surface, and that the intensity $\mathcal{F}$ makes an angle, θ, with this surface. The work done in moving a unit of electricity from a to b is

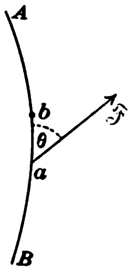


FIG. 195.

$$V_a - V_b = \mathcal{F} \cos \theta \cdot ab; \tag{4}$$

but, by definition, $V_a = V_b$; therefore,

$$\cos \theta = 0,$$

or the force is perpendicular to the surface.

278. Field Due to a Charged Sphere. — If one of the equipotential surfaces, B (Fig. 194), were to be transformed into a conducting shell, it is evident that the field would be unaltered, for B is still an equipotential surface and of the same numerical value, since its potential is produced by A alone. Likewise the potential of every other part of the field is the same as before. Also, since the equipotential surfaces are at the same distance, the intensity at every point is unchanged. If, now, A be connected by a wire to B, the lines of force within the conductor will be wiped out, but

the external field will remain the same, as was shown by Faraday's ice-pail experiment. Therefore, a charged spherical conductor exerts the same force without its surface as it would were the electricity collected at its center.

279. Field Due to Two Unlike Charges. — Fig. 196 shows the field produced by a positive charge of 10 units at A and

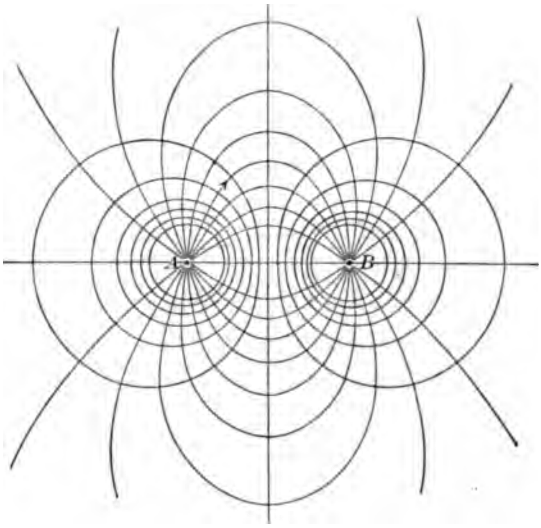


FIG. 196.

an equal negative charge at B. Near the points the equipotential surfaces are practically spherical, but further out they become flattened on the side toward the opposite charge. The two systems of surfaces are separated by a plane, which is a surface of zero potential.

Any one of the equipotential surfaces about A might be regarded as the surface of a conductor charged with 10 units of electricity, under the influence of an equal opposite charge

than in its absence, as if it had a positive charge on that side. Regarding the lines of force as elastic bands, it appears that the horizontal component of forces on the left will slightly exceed those on the right, or A will experience a resultant force, as though B attracted it.

282. Potential Due to a Charged Point. — Let q be the charge at the point A (Fig. 199). The intensity at the point p_1 at a distance, r_1, from A will be $F_1 = \frac{q}{Kr_1^2}$, and, similarly, that at p_2 will be $F_2 = \frac{q}{Kr_2^2}$. By definition the change in potential between these points is the work which would be done in carrying a unit of electricity from p_2 to p_1. If the intensity were constant between these points, this work could be calculated at once by multiplying the force by the distance.

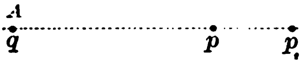


Fig. 199.

In the case where the force varies inversely as the square of the distance, it may be shown that the work is obtained by multiplying the distance by the geometric mean value of the force between the points. Thus,

$$(5) \qquad V_1 - V_2 = \frac{q}{Kr_1r_2}(r_2 - r_1) = \frac{q}{K}\left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$

If the point p_2 is at infinity,

$$(6) \qquad V_1 = \frac{q}{Kr_1},$$

or the potential due to a single charge is equal to the quotient of the charge by the specific inductive capacity and the distance of the point considered.

283. Behavior of a Dielectric Body in an Electric Field. — If a dielectric sphere is brought into a field produced by a

charged point, A, in a medium whose specific inductive capacity is less than its own, there will be a distortion of the field like that represented in Fig. 200. The equipotential surfaces are crowded away from the sphere, as in Fig. 198, though now some of the surfaces pass through it. It may be observed that the distances between surfaces within the sphere are greater than they would be in the outside medium, showing that the intensity is less. The distortion of the equipotential surfaces has lowered the potential of points near the left side of the sphere, but raised those at the right side, as would be the case if there were a negative charge on the left side of the sphere and a positive charge on the right. This inductive action of the point A, producing an apparent charge on the sphere, is also indicated by the crowding of the lines of force into the material of the sphere, and by the shortened distance between the equipotential surfaces to the right and to the left of it.

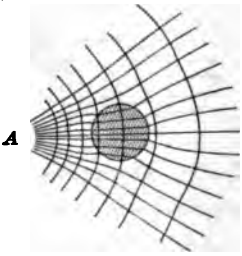


FIG. 200.

By observing the change in the angle at which the lines of force enter and leave the surface, it will appear that the sphere ought to be drawn toward the point A, since the lines on the left would have a greater horizontal resultant force than those on the right. This important conclusion may be differently stated thus: A dielectric body, placed in an electric field in a medium whose dielectric constant is less than its own, experiences a force urging it toward that part of the field where the intensity is a maximum.

In a uniform field it would experience no force of translation, but an elongated body would set itself so that its length would be parallel to the lines of force, as is obviously

upon a similar conducting surface surrounding *B*, by simply expunging all the lines within these surfaces.

It will be noticed that the intersections of the lines and the equipotential surfaces are orthogonal. If the lines of force be regarded as so many stretched elastic bands connecting the bodies, the mechanical force exerted would be such as to draw the bodies nearer together; that is, they show an apparent attraction.

280. Field Due to Two Like Charges. — The electric field produced by two like but unequal charges at the points *A* and *B* is shown in Fig. 197. The lines of force form two

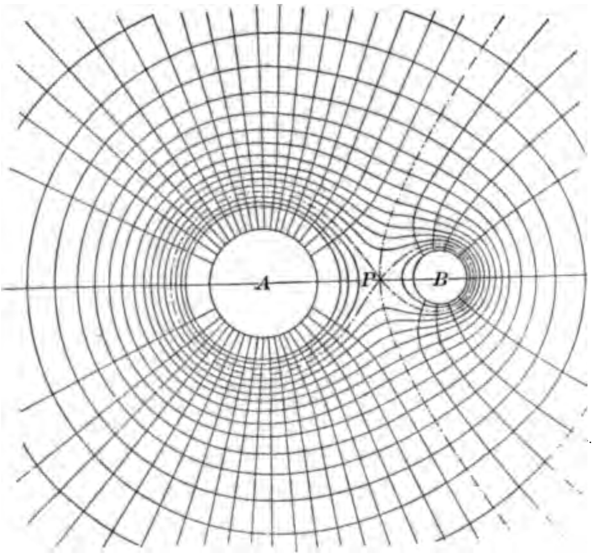


FIG. 197.

independent systems starting from each point and running to oppositely charged surfaces supposed to be at a great distance.

The two systems are separated by a surface resembling one of the sheets of a hyperboloid of two sheets. A system of equipotential surfaces encloses each charged point, and these are in turn surrounded by a third system, of which the limiting case is a surface of two lobes meeting at the point *P*, where the force vanishes. If, as before, the lines of force be regarded as elastic bands, the two charged points would be drawn apart; that is, they show an apparent repulsion.

281. Field about an Insulated Conductor. — Fig. 198 illustrates roughly the inductive effect of a charged body, *B*, on a spherical conductor, *A*. As the conductor is brought into the field, it assumes the potential at that point, and the equipotential surface includes *A* as a cavity in itself. Since as many lines of force leave the ball on one side as enter it upon the other, *A* has not a definite charge in the usual sense of the term, but only an apparent negative charge on the left side and an apparent positive charge on the right. This inductive action is also clearly shown by the arrangement of the equipotential surfaces. On the left of *A* they have been displaced, so that there is a more rapid fall of potential than in the undisturbed field; that is, the force measured at any point between *B* and *A* will be greater on account of the presence of *A*, as if the latter carried a negative charge on the left side. Similarly, the potential of points just to the right of *A* is greater

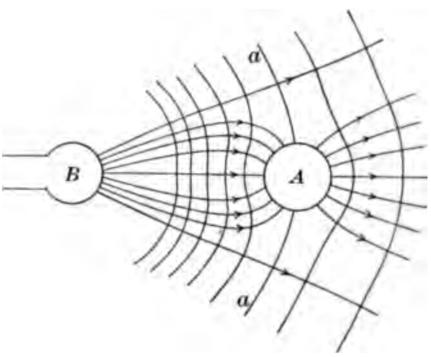


FIG. 198.

than in its absence, as if it had a positive charge on that side. Regarding the lines of force as elastic bands, it appears that the horizontal component of forces on the left will slightly exceed those on the right, or A will experience a resultant force, as though B attracted it.

282. Potential Due to a Charged Point. — Let q be the charge at the point A (Fig. 199). The intensity at the point p_1 at a distance, r_1, from A will be $F_1 = \frac{q}{Kr_1^2}$, and, similarly, that at p_2 will be $F_2 = \frac{q}{Kr_2^2}$. By definition the change in potential between these points is the work which would be done in carrying a unit of electricity from p_2 to p_1. If the intensity were constant between these points, this work could be calculated at once by multiplying the force by the distance. In the case where the force varies inversely as the square of the distance, it may be shown that the work is obtained by multiplying the distance by the geometric mean value of the force between the points. Thus,

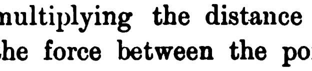


FIG. 199.

$$(5) \qquad V_1 - V_2 = \frac{q}{Kr_1r_2}(r_2 - r_1) = \frac{q}{K}\left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$

If the point p_2 is at infinity,

$$(6) \qquad V_1 = \frac{q}{Kr_1},$$

or the potential due to a single charge is equal to the quotient of the charge by the specific inductive capacity and the distance of the point considered.

283. Behavior of a Dielectric Body in an Electric Field. — If a dielectric sphere is brought into a field produced by a

charged point, A, in a medium whose specific inductive capacity is less than its own, there will be a distortion of the field like that represented in Fig. 200. The equipotential surfaces are crowded away from the sphere, as in Fig. 198, though now some of the surfaces pass through it. It may be observed that the distances between surfaces within the sphere are greater than they would be in the outside medium, showing that the intensity is less. The distortion of the equipotential surfaces has lowered the potential of points near the left side of the sphere, but raised those at the right side, as would be the case if there were a negative charge on the left side of the sphere and a positive charge on the right. This inductive action of the point A, producing an apparent charge on the sphere, is also indicated by the crowding of the lines of force into the material of the sphere, and by the shortened distance between the equipotential surfaces to the right and to the left of it.

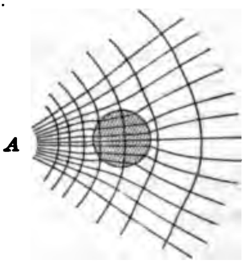


FIG. 200.

By observing the change in the angle at which the lines of force enter and leave the surface, it will appear that the sphere ought to be drawn toward the point A, since the lines on the left would have a greater horizontal resultant force than those on the right. This important conclusion may be differently stated thus: A dielectric body, placed in an electric field in a medium whose dielectric constant is less than its own, experiences a force urging it toward that part of the field where the intensity is a maximum.

In a uniform field it would experience no force of translation, but an elongated body would set itself so that its length would be parallel to the lines of force, as is obviously

seen to be the result of regarding the lines of force in Fig. 201 as under tension. If, however, such an elongated body were placed in a non-uniform field where the intensity varied greatly within a small space, the expulsion effect, being different for different parts of the body, might be great enough to overbalance the forces which would set its length parallel to the lines of force, and cause it to set transversely with respect to these lines.

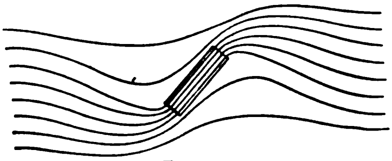


FIG. 201.

In the case where the value of K is greater in the surrounding medium, the distortion of the field is similar to Fig. 202.

The equipotential surfaces are crowded closer together within the sphere, indicating that the intensity is greater at a point inside the sphere than it would be if the medium were all such as that outside. The potential of points just to the left of the sphere has been raised, and that of points on the right lowered, as if there was a positive charge placed on the side toward A and a negative charge on the side away from A. Regarding the lines of force as elastic bands, the distortion of the field is seen to have increased the angle with the horizontal where the lines meet the sphere on the left, but decreased it on the right. The horizontal component of the forces represented by the lines on the right will be in excess, and the sphere will be drawn away from A. This result may be otherwise stated thus: A dielectric body, placed in an electric field in a medium whose dielectric constant is greater than its own,

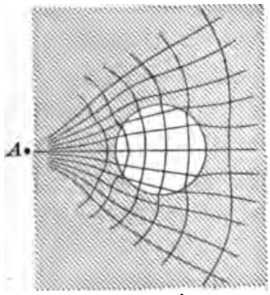


FIG. 202.

experiences a force urging it toward that part of the field where the intensity is a minimum.

In a uniform field the body would experience no force of translation, but an elongated body having a value of K less than the outside medium would set itself parallel to the lines of force for reasons similar to those given in connection with Fig. 201. If, however, such an elongated body be placed in a non-uniform field in which the change of intensity is very great, the expulsion effect, different at different parts of the body, might be great enough to overbalance what has been referred to as tension along the lines of force, and set the length of the body transversely with respect to these lines.

EXAMPLES.

1. A spherical conductor 8.6 cm. in diameter is charged with 65 units of electricity. What will be the potential at a distance of 75 cm. from the surface of the sphere? *Ans.* .82 $\text{cm.}^{\frac{1}{2}}\ \text{gm.}^{\frac{1}{2}} / \text{sec.}$

2. If a charge of 27 units is placed at one of the corners of an equilateral triangle, the length of each of whose sides is 32 cm., and a charge of 68 units at the second, what will be the intensity at the third corner of the triangle? *Ans.* 0.083 $\text{gm.}^{\frac{1}{2}} / \text{cm.}^{\frac{1}{2}}\ \text{sec.}$

3. If charges $q_1, -q_2$ are placed at points separated a distance, d, show that the surface of zero potential is a sphere of radius, $\frac{q_1 q_2}{q_1^2 - q_2^2} d$, with its center at a distance, $\frac{q_1^2}{q_1^2 - q_2^2} d$, from q_1.

CHAPTER XVIII.

ELECTROSTATIC INSTRUMENTS.

284. Electrical Machines. — Machines designed to generate charges of electricity at high differences of potential are of two types: 1°, Frictional Machines, or those in which the electrification is produced and augmented by the rubbing of two dissimilar bodies; 2°, Influence Machines, or those in which a given electrification is used to generate other charges by induction.

In machines of the first type a cylinder or a circular plate of glass is revolved so as to rub against a surface of leather coated with an amalgam of zinc, by which process the glass becomes positively and the rubber negatively electrified. The charge on the glass is collected on suitable conductors and may be used for any desired purpose.

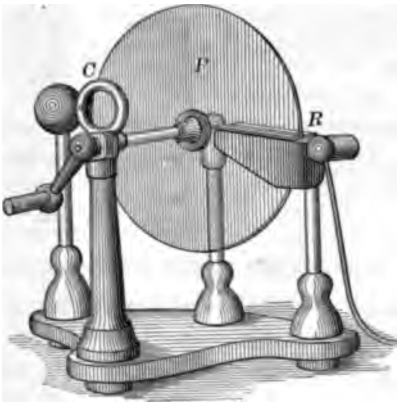


Fig. 203.

One of the best forms of the friction machine is shown in Fig. 203. P is the glass plate, which is electrified by revolving it between the rubbers at R. The charge is collected by a pair of rings at C, which are set on the sides toward the plate with a series of sharp points. On account of the low efficiency this type of machine has now fallen into disuse.

285. Electrophorus. — The first of the induction machines was invented by Volta in 1775, and is known as the electrophorus. It consists of a metal plate, *C* (Fig. 204), with an insulating handle, by which it may be raised or replaced on the base of the apparatus, a resinous disc, *A*, backed by the metal sole *B*. The disc having been electrified by rubbing with cat's fur, the plate is brought very close to the base or set upon it.

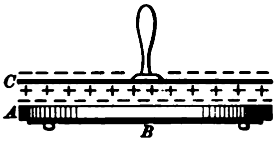
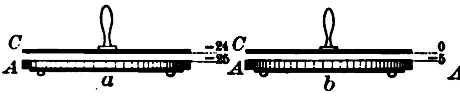


FIG. 204.

Contact in this case has no perceptible effect on the charge of *A*, since the plate really touches the resin in but a few points which are themselves non-conducting. The electrical state of *C* is then that represented in the figure. The top of *C* is next discharged to the ground, or the walls of the room, by touching it with the finger. If the plate be now raised, the electricity, which was strongly attracted by the negative charge of *A*, will be distributed over the conductor, and, on presenting the knuckle, will be discharged with an accompanying spark. As no part of the original charge is wasted by this process, it may be repeated any number of times. The sole *B* is not essential to the successful working of the apparatus, but may retard the slow leakage of the resinous charge from the base.



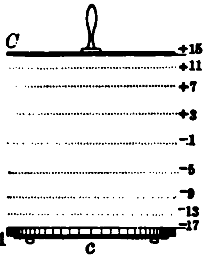


FIG. 205.

The process of charging by induction, exemplified in the electrophorus, may be explained in a more philosophical way, and in accordance with the modern view of electrical phe-

nomena, by use of the term potential. Thus, suppose that the base A (Fig. 205a) is originally at a negative potential represented, for example, by 25 units, then as the plate C is approached to A its potential will fall almost to that of A, say to -24. If C is now touched to earth, it will take the potential of the latter, which is taken arbitrarily as zero. Also, since C is very near to the base, the potential of the latter will rise, and the state of things may be that represented in Fig. 205b. Finally, if C is lifted off from the base, the potential of the former will rise and the latter fall in a way not unlike that shown in Fig. 205c. As there is a tendency for the electricity to run off from the plate to the earth, C now has a positive charge in the ordinary sense of the term. It is to be noticed that the electrical energy obtained in the cover is at the expense of the work done against the electrical forces in raising it from the base.

286. Reciprocal Electrophorus. — An arrangement of electrophoruses, by which a very small initial charge may be mul-

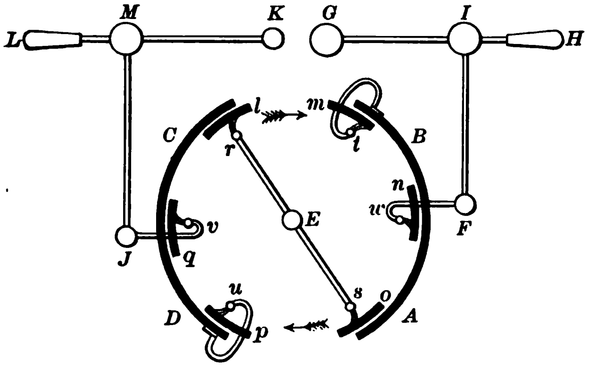


FIG. 206.

tiplied by work done in rotation, is shown in Fig. 206. AB and CD are two fixed plates, called inductors. l, m, n, o, p, q

are a series of metal bodies, known as carriers, carefully insulated from each other and so arranged that they may be revolved in the direction of the arrow about the axis E. r and s are two stationary tinsel brushes, called connectors, which communicate with the ground through the axis E. t, u are another pair of brushes, called replenishers, by which the inductors are, for a portion of the revolution, brought in contact with the carriers. v and w are collecting brushes designed to transfer the charges from the carriers to the conductors IG and MK. In order to start the machine, a small charge must be given to one of the inductors, C. Suppose that it is positive; it will induce negative electrification on the nearer side of the carrier l, and on the farther side positive electrification, which is, however, at once neutralized by the brush r in communication with the earth. Accordingly, as l moves away from C, it carries a negative charge, and when it comes into the position m, this is shared with BA. At the position n the charge still remaining on it is given up to the conductor FIG. In a similar manner the carrier in the position of o will take away a positive charge, a part of which is given to DC, increasing the original electrification of the inductor, and the remainder is deposited on JMK. Thus, the positive charge on K and the negative one on G increase until a spark passes, or the leakage puts a stop to further accumulation. It may be remarked that Fig. 206 is rather a diagrammatic representation of a prototype of all influence machines than a form of apparatus designed for actual use. One of the earliest machines embodying these principles was the *revolving doubler* invented by Nicholson in 1788.

287. Toepler-Holtz Machine. — In a form of machine invented by Holtz, though somewhat modified by Toepler and

Voss, the construction is but little varied from the diagram of Fig. 206.

The inductors are two pieces of tin foil, *AB*, *CD*, pasted to the back of a stationary circular glass plate, *P* (Fig. 207).

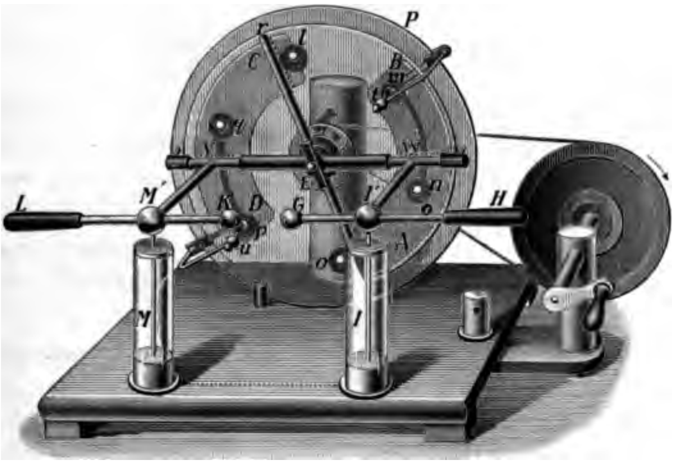


Fig. 207.

The carriers are discs of tin foil, fastened to another glass plate, which revolves just in front of *P*. The collecting brushes of Fig. 206 are replaced by combs with a number of sharp-pointed teeth set very close to the revolving disc, but so that there is no metallic contact between the collecting system and the carriers. Combs are also used in conjunction with the brushes *r* and *s*, to remove the charge induced on the front of the revolving plate. The capacity of the collecting system is increased by the introduction of two small Leyden jars. The operation of this machine is in no way different from that given in Art. 286.

288. Holtz Machine. — In the form of machine which was invented by Holtz in 1865, and usually bears his name, the

inductors are pieces of paper, *AB*, *CD*, pasted on a stationary plate near two holes or windows, *Q*, *N*, through which project small tongues of paper, *t*, *u* (Fig. 208), connected

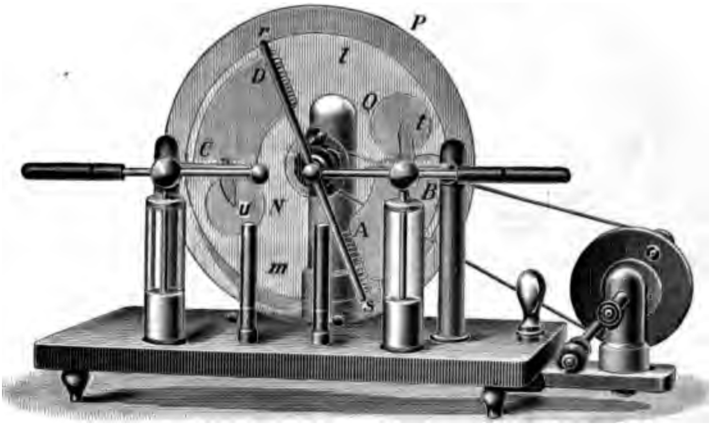


FIG. 208.

with the inductors. These points serve the purpose of replenishers, but the charge is transferred from the plate to the inductor without direct contact. The distinctive feature of the original Holtz machine, and the one to which it owes its great efficiency, is the entire absence of metal carriers, their function being performed by the revolving glass plate *l*, *m*. This form of machine is slow in starting and does not work well in a humid atmosphere, but under reasonably favorable conditions the difference of potential obtainable with it exceeds that of any other yet devised.

289. Wimshurst Machine. — In the Wimshurst machine two glass plates, carrying a number of tin-foil sectors, are arranged so as to revolve with the same speed in opposite directions. A neutralizing brush is arranged before each plate and at right angles to each other, but the collecting

system consists of combs similar to those used on other machines. The distinctive peculiarity of the Wimshurst machine is that the metal sectors are made to serve alternately as inductors and carriers. Its operation may be readily understood from an inspection of Fig. 209. The carrier at *a* is acting as inductor on *a′*, but when they reach the positions *b*, *b′*, their functions are reversed. The great advantage of this machine is that it will work in a very damp atmosphere. A picture of the working machine is shown in Fig. 210.

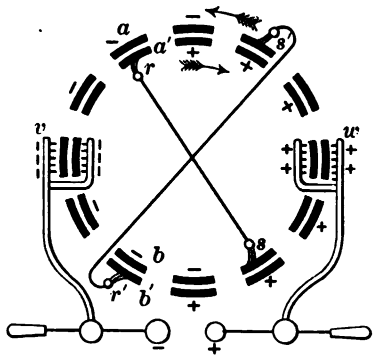


Fig. 209.

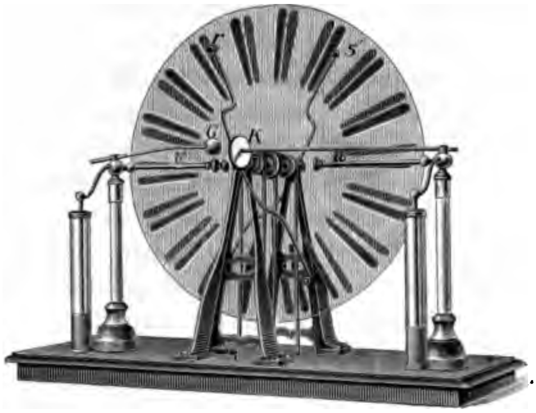


Fig. 210.

290. Hankel's Electrometer. — A modification of the gold-leaf electroscope, which permits the comparison of potentials

to a certain degree of approximation, is shown in Fig. 211. *A* and *B* are two conducting plates between which a constant difference of potential is maintained by means of a water battery (Art. 341). *C* is a gold leaf suspended from the insulated knob *D*, which may be connected by means of a wire with the point whose potential it is desired to examine. Under these circumstances, the gold leaf is deflected toward the electrified plate differing most from its own potential a distance which is nearly proportional to this difference. This particular form of instrument is known as Hankel's electrometer.

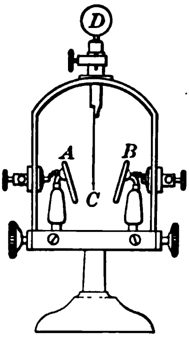


FIG. 211.

291. Quadrant Electrometer.—Fig. 212 shows an important type of instrument, devised by Lord Kelvin for the measurement of small differences of potential, and called the quadrant electrometer. It consists essentially of a metallic cylindrical box divided into four segments, *A*, *A'*, *B*, *B'*, within which is hung, by a double thread, *t*, *t'*, a light aluminum blade or needle, *CD* (Fig. 213). The bifilar suspension furnishes a small moment of restitution, which restores the needle, after a torsional displacement, to a position symmetrical with respect to the quadrants. The opposite

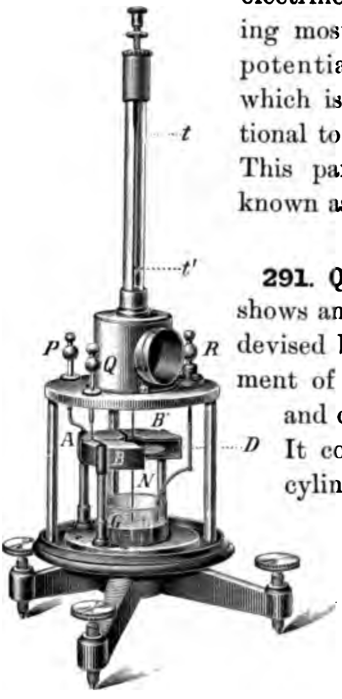


FIG. 212.

quadrants AA', BB' are connected by the wires. A wire, N, attached to the lower side of the needle dips in a dish of sulphuric acid, which serves the threefold purpose of damping the vibration of the needle, making electrical connection between the binding post R and the needle, and absorbing the aqueous vapor in the case with which the electrometer is always enclosed when in use. If a constant potential difference is maintained between each pair of quadrants by means of a condenser or battery and the needle is charged successively to different potentials, the angular deflections of the needle, which may be read by aid of a telescope and scale, will be proportional to the changes in potential.

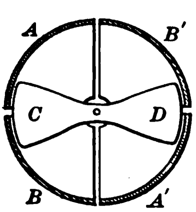


FIG. 213.

The instrument may also be used by charging the needle and one pair of quadrants to the same potential, while the other pair is kept at a different potential. In this case the deflection of the needle is proportional to the square of the potential difference between the quadrants.

292. Capillary Electrometer. — When the surface between mercury and sulphuric acid is electrified, or, more properly, polarized, the surface tension is considerably affected, apparently by the release of one of the ions at the surface.

This phenomenon has been utilized by Lippmann in the construction of a capillary electrometer, which has a limited application to the measurement of potential differences less than a volt. One form of the apparatus is shown in Fig. 214.

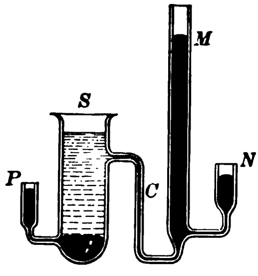


FIG. 214.

C is a capillary tube connected on one side with a tube, M, filled with mercury, and on the other side with the tube S, containing dilute sulphuric acid. Platinum wires sealed into M and S, and terminating in the mercury cups P and N, serve as electrodes. On connecting P to the positive and N to the negative pole of a small battery, the thread of mercury in the tube C will fall a small distance, which may be taken as proportional to the potential difference, provided this does not approach an amount sufficient to produce continuous electrolysis. The variations in the hight of the mercury are best read by the aid of a micrometer microscope.

293. Capacity.—The electrostatic capacity of a conductor is defined as the quotient of its charge by its potential. If the charge be denoted by Q, the potential by V, and the capacity by C, then

$$C=\frac{Q}{V}.$$

The value of C depends only on the form and dimensions of the conductor and on the nature of the dielectric.

294. Condenser.—The capacity of an insulated conductor is greatly increased by the presence of another conductor connected to the earth or to the walls of the room. Thus, let A (Fig. 215) be a conducting sphere, which is charged to a positive potential, say,

$$V=15.$$

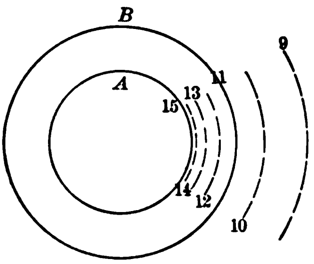


Fig. 215.

If equipotential surfaces be drawn about A for unit potential differences, there will be fourteen such surfaces between the conductor and the walls of the room. Now, let A be

surrounded with a conducting sphere, B. It will take the potential of the surface with which it coincides without altering the field. If the outer sphere be now connected to the walls of the room, its potential will fall to zero, and that of A to $V=4$, so that if A be again put in communication with a reservoir of electricity maintained constantly at the potential 15, it is evident that a large quantity of electricity will have to be added to A before its potential will again rise to 15, because the presence of B at zero keeps the potential of A below what it would be were A alone in the field. This arrangement of conductors is called a *condenser*.

295. Specific Inductive Capacity. — If the space between A and B in Fig. 215 were occupied by any other insulator, the potential of A would in general be lowered, or, what amounts to the same thing, the capacity of the condenser would be increased.

The original discovery of this fact was made by Cavendish, but not published. It was independently discovered by Faraday in 1837, and became the starting point of the modern theory which assigns the electrification to the state of the dielectric rather than to the condition of the conductor.

The ratio of the capacity of a condenser where the dielectric is a given substance to the capacity where the dielectric is air was called by Faraday the *specific inductive capacity* of the substance. The values of this quantity, which is identical with the dielectric constant of Art. 271, were investigated for a number of substances by comparing the capacities of a spherical condenser similar to that of Fig. 215, when different dielectrics were used; but Faraday's results are now recognized as too small. The difficulties in the way of securing precise values of the specific inductive capacities of solids have never been overcome. It is found that, when a

condenser with solid dielectric is charged, its capacity appears to increase with the time, a phenomenon sometimes described as electric absorption or soaking in. The explanation seems to be that there is a slow yielding of the insulator under the electric stresses to which it is subjected.

The following tables show the approximate values of K found by various investigators.

TABLE OF SPECIFIC INDUCTIVE CAPACITIES.

SOLIDS.

SUBSTANCE.	SPECIFIC INDUCTIVE CAPACITY.	INVESTIGATOR.
Hard crown glass . .	6.96	Hopkinson
Very light flint glass .	6.61	"
Light flint glass . .	6.72	"
Dense flint glass . .	7.38	"
Extra dense flint glass	9.90	"
Plate glass	8.45	"
"	6.10	Wüllner
"	5.83 to 6.34	Schiller
Shellac	2.95 to 3.73	Wüllner
Paraffine	1.96	"
"	2.32	Boltzmann
"	1.85 to 2.47	Schiller
Ebonite	2.56	Wüllner
"	3.15	Boltzmann
"	2.21 to 2.76	Schiller
Sulphur	2.88 to 3.21	Wüllner
"	3.84	Boltzmann

LIQUIDS.

Ether	4.2 to 4.4	Benzol	2.36
Carbon disulphide .	2.58	Petroleum . . .	2.02 to 2.07

GASES.

Air	1.000	Hydrogen	0.9998
Carbon dioxide . . .	1.0008	Sulphur dioxide . . .	1.0037
Coal gas	1.0004	Vacuum	0.9985

296. Leyden Jar. — The Leyden jar is a convenient form of condenser, consisting of a glass jar (Fig. 216) coated inside and out to a certain hight, G, with metal foil. The inner coat of the jar is connected to the knob D by means of the chain B and the rod C, inserted in the wooden stopper H.

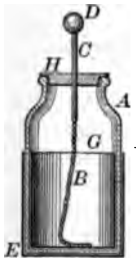


FIG. 216.

To charge the jar it is only necessary to set it upon a table or other uninsulated support and connect the knob with one of the terminals of an electrical machine. The discovery of the principle of the condenser appears to have been made by Kleist, a German monk, in 1745, who showed that, if a nail be thrust through the cork of a vial in which there was a small quantity of mercury, the bottle could be very strongly electrified.

A similar discovery was made the following year by Muschenbroek, of Leyden, while testing a notion he had, that water enclosed in a glass vessel would not lose its charge so rapidly as an exposed conductor.

A wire connected with one terminal of the machine was arranged so as to dip into a vessel of water held in the hand of a pupil named Cuneus. The latter, attempting to remove the charging wire after the water had been electrified, received a violent shock. This experiment attracted wide attention and was repeated in many laboratories. The name Leyden jar came to be applied to this particular form of condenser from the city in which the experiment originated.

297. Capacity of a Spherical Conductor. — Since the external field due to a charged sphere is in every respect equivalent to that produced by the same charge placed at its center, the potential of a charge, q, placed on a sphere of radius, a, must be, by equation 6, Art. 282,

$$V = \frac{1}{K}\frac{q}{a}, \tag{1}$$

whence the capacity of the sphere is

$$C = \frac{q}{V} = Ka; \tag{2}$$

that is, the radius of the sphere multiplied by the specific inductive capacity of the surrounding medium.

298. Capacity of a Spherical Condenser. — Let A and B (Fig. 217) be two concentric spherical conductors having radii a and b; and suppose that A is charged with a quantity of electricity, q. Then, by equation 5, Art. 282, the difference of potential between A and B is

$$V_a - V_b = \frac{q}{K}\left(\frac{1}{a} - \frac{1}{b}\right). \tag{3}$$

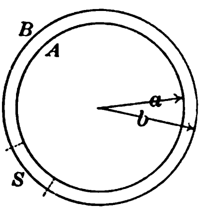


FIG. 217.

If B be now connected to earth, its potential will become zero, and that of A will fall, keeping, however, the same difference between B and A. The potential of A thus becomes

$$V_a = \frac{q}{K}\cdot\frac{b-a}{ab}.$$

Hence, the capacity of A, under these circumstances, is

$$C = \frac{q}{V_a} = K\frac{ab}{b-a}. \tag{4}$$

From this equation it appears that the capacity of such a condenser may be made very great by diminishing the distance between the surfaces, and also if $b - a$ is small, compared to the dimensions of the spheres, the capacity is nearly proportional to the square of the radius of the condenser.

The value of C may be written

(5)
$$C = K\frac{(a^2 - a^2 + ba)}{b - a} = K\left(\frac{a^2}{b - a} + a\right) = K\left(\frac{A}{4\pi d} + a\right),$$

where A denotes the area of the inner sphere and d the distance between the shells. It thus appears that, by the addition of the outer shell connected to earth, the capacity of the sphere has been increased by the large quantity $\frac{KA}{4\pi d}$.

299. Capacity of a Parallel Plate Condenser. — Imagine that a portion of the spherical condenser (Fig. 217), having an area, S, is cut away. It is obvious that the capacity c of this portion will be the same part of C that S is of A, or,

$$\frac{c}{C} = \frac{S}{A}.$$

Substituting the value of C from equation 4,

(6)
$$C = \frac{KS}{4\pi a^2}\,\frac{ab}{b - a} = \frac{KS}{4\pi\,(b - a)} \cdot \frac{b}{a}.$$

In case the conducting plates are planes, $\frac{b}{a} = 1$; thus letting d denote their separation,

(7)
$$C = \frac{KS}{4\pi d}.$$

EXAMPLES.

1. If a sphere 6.4 cm. in diameter and charged with 98 C. G. S. units be connected by a fine wire with a second conducting sphere whose radius is 2.5 cm., what will the charge and potential of each sphere then be?

Ans. $\begin{cases} V = 17.2 \text{ gm.}^{\frac{1}{2}} \text{ cm.}^{\frac{1}{2}} / \text{sec.} \\ q_1 = 55 \text{ gm.}^{\frac{1}{2}} \text{ cm.}^{\frac{3}{2}} / \text{sec.} \\ q_2 = 43 \text{ gm.}^{\frac{1}{2}} \text{ cm.}^{\frac{3}{2}} / \text{sec.} \end{cases}$

2. Three spheres having capacities 3 cm., 5 cm., and 8 cm. are charged to potentials 30, 24, and 18 units, respectively, and then connected by a fine wire. What will be their common potential?

Ans. 4.9 cm.$^{\frac{1}{2}}$ gm.$^{\frac{1}{2}}$ / sec.

3. A spherical conductor having a radius of 2.7 cm. is charged to a potential of 125 C. G. S. units. On being brought in contact with a second uncharged conductor, the potential fell to 22 units. What was the capacity of the second body? *Ans.* 12.7 cm.

4. A spherical conductor having a radius, r_1, and a potential, V_1, is introduced into a second hollow sphere having a radius, r_2, and a potential, V_2, and touched to it. Required the charge and the potential of each sphere after contact.

Ans. $V = r_1 V_1 + r_2 V_2 / r_2$. $q_2' = r_1 V_1 + r_2 V_2$.

5. Two small bodies having capacities C_1, C_2, and potentials V_1, V_2, are connected by a fine wire. At what point between the bodies will the potential remain unchanged? *Ans.* Half-way.

6. An air condenser charged to a potential, V, is brought into contact with an exactly similar condenser with an unknown dielectric. If the final potential is V', what is the specific inductive capacity of the dielectric? *Ans.* $K = (V - V') / V'$.

7. A sphere of 2 cm. radius is charged with 80 units of electricity and placed at a given distance from a charged sphere of 3 cm. radius. On connecting the spheres with a fine wire, the force between them is found to be unaltered. What was the charge on the second sphere?

Ans. 53.6 or 119 gm.$^{\frac{1}{2}}$ cm.$^{\frac{3}{2}}$ / sec.

8. What is the capacity of a spherical condenser having an external diameter of 78 cm. and a dielectric 0.56 cm. thick, with a specific inductive capacity of 4.9? *Ans.* $0.536(10)^5$ cm.

9. A cylindrical Leyden jar having a thickness of 0.16 cm. and a diameter of 8.4 cm. is coated with tin foil on the bottom and on the sides, to a hight of 10 cm. What is the capacity of the jar, taking the dielectric constant of the glass as 5.5? *Ans.* 870 cm.

10. Calculate the energy of a condenser having a capacity of 950 cm. charged to ~~a potential~~ 625 C. G. S. units. *Ans.* 206 ergs.

11. A charged conductor having a capacity of 67 cm., at a potential of 85 units, is made to share its charge with an unelectrified body, whose capacity is 125 cm. How much energy is lost by the discharge? *Ans.* $1.58(10)^5$ ergs.

12. If a spherical air condenser consisting of shells 64 cm. and 60 cm. in diameter be charged to a potential of 500 C. G. S. units, and afterward connected to a similar uncharged condenser having radii 64 cm. and 58 cm., how much energy will be lost by the change? *Ans.* $0.404(10)^7$ ergs.

13. A condenser consists of two circular plates 15 cm. in diameter, separated by a film of air 1.2 millimeters thick. When 750 ergs have been expended in charging it, what will be the potential difference between the plates and the charge?

Ans. $V = 3.58$ gm.$^{\frac{1}{2}}$ cm.$^{\frac{1}{2}}$ / sec. $Q = 419$ gm.$^{\frac{1}{2}}$ cm.$^{\frac{3}{2}}$ / sec.

14. Two insulated spheres having radii 4.8 cm. and 1.6 cm. are connected by a fine wire and have 148 C. G. S. units of electricity shared between them. Required the charge on each sphere and their potential.

$$\textit{Ans.} \quad \begin{cases} V = 23.1 \text{ cm.}^{\frac{1}{2}} \text{ gm.}^{\frac{1}{2}} / \text{sec.} \\ Q_1 = 111 \text{ cm.}^{\frac{3}{2}} \text{ gm.}^{\frac{1}{2}} / \text{sec.} \\ Q_2 = 37 \text{ cm.}^{\frac{3}{2}} \text{ gm.}^{\frac{1}{2}} / \text{sec.} \end{cases}$$

CHAPTER XIX.

THE ELECTRIC DISCHARGE.

300. Disruptive Discharge. — When the intensity at any point of a dielectric is gradually increased, a limit will be finally reached, beyond which the medium is unable to resist the stress exerted on it, and a sudden discharge occurs, accompanied by a flash of light and a cracking noise. In fluids the rupture produced by the discharge is self-healing, but in solids the break is, of course, permanent. Leyden jars are frequently spoiled by charging too highly, and forcing a discharge through the glass where the resistance is least. As the striking distance between two conductors is increased, the path of the spark (Fig. 218) departs widely

FIG. 218.

from a straight line, assuming the characteristic broken appearance of the lightning flash. The explanation is that the fracture follows the line of least resistance, its position probably being determined by the presence of dust in the air, much as a piece of paper when torn in two breaks along a line of weakest points.

When the discharge takes place through a piece of cardboard, a burr is found to be raised on both sides. The

phenomenon is to be accounted for by the explosive rupture of the dielectric. If, for instance, a piece of fine wire were to be passed through a sheet of paper, and then suddenly expanded, the rent thus made would be not unlike the hole pierced by the electric spark. The duration of the discharge is exceedingly short, as may be shown by using the spark to illuminate a disc covered with alternate sectors of black and white. If the disc be whirled so rapidly that it appears a uniform gray, the flash of the electric spark will show each sector almost as distinctly as if the disc were at rest. By observing the spark in a revolving mirror, which would have the effect of drawing a luminous point into an extended track if the illumination occupied a sensible time, Wheatstone showed that the whole discharge must take place in less than $\frac{1}{20000}$ of a second.

The appearance of the spark at the two electrodes is somewhat different. If, for instance, the spark is branched, these branches always point toward the negative electrode. If the electrodes are spheres of unequal size, the spark length is greater when the negative pole has the larger diameter, especially if the spark is accompanied by some other form of discharge.

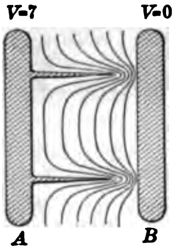


FIG. 219.

301. Discharge from Points. — Suppose that A (Fig. 219) is a conductor possessing several points, and charged, say, to a potential, $V = 7$. Let B be another conductor maintained at the potential zero, and draw surfaces for unit difference of potential. It is then evident, from the crowding together of the surfaces in the vicinity of the points, that the electric intensity is greatest in those regions. If, now, the space between the conductors be filled with air associated with particles of

dust or aqueous vapor, these particles will first become electrified by contact, and then urged across the intervening space with considerable force. The difference of potential between the conductors can thus be rapidly reduced by electric convection.

If a tourniquet, such as is shown in Fig. 220, be placed on two parallel insulated wires, and highly electrified, the repulsion between the points and the electrified particles in the air will be sufficiently great to set the wheel in rapid rotation.

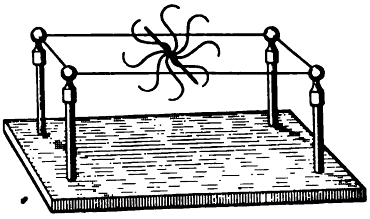


FIG. 220.

The phenomenon of discharge from points was first observed by Franklin, and applied by him to the protection of buildings by the erection of lightning rods terminating in several sharp points. It is evident that all points should be eliminated from conductors designed to carry charges.

302. The Brush Discharge. — When a rounded conductor is highly charged and approached to some other conductor at lower potential, the discharge takes a form known as the electric brush, and is accompanied by a peculiar hissing sound. In a darkened room the ball is seen to be terminated by a brush of pale blue light consisting of innumerable twig-like ramifications diverging as they leave the metal. In the brush discharge the electricity seems to be carried in part by particles of metal torn from the electrodes. The brush is more easily formed at the negative than at the positive electrode, at the same time appearing smaller and less finely divided. If the rounded end of the conductor be replaced by a sharp point, the discharge takes the form mentioned in Art. 301, showing in the dark a quiet and

continuous glow, sometimes separated from the conductor by a dark space.

303. Oscillatory Discharge. — When a condenser is discharged by the sudden rupture of the dielectric, the potential does not, in general, assume its final value at once, but oscillates through it much as an elastic system, when displaced, will vibrate through its position of rest. That the discharge is oscillatory in certain cases may be shown by the following experiment.

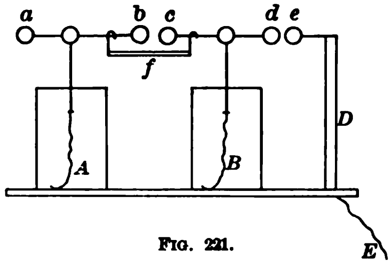


FIG. 221.

Let A and B (Fig. 221) be two Leyden jars, furnished with sliding rods, by which the spark gaps bc and de may be adjusted at pleasure. Let the outer coatings of the jars and the ball e be connected to the earth.

At first let b and c be connected by a conducting wire, and having arranged d at some convenient distance, say a centimeter from e, let the jars be charged till a spark passes at de. In this arrangement both jars are discharged at once, and no spark appears at bc. Next let bc be connected by a glass tube moistened within, and the jars charged as before till a spark passes at de. At the same time a spark will be observed at bc, which by adjustment of the gap may be made almost twice as long as that at de. As the moistened tube is a poor conductor and the process of charging is a comparatively slow one, the potential of both jars at the moment of discharge must have been sensibly the same, say V. But the potential difference at bc was evidently greater than that at de, because the spark length was greater. This difference may be accounted for by supposing that, at the moment of

discharge between d and e, the potential of B fell not only to zero, but nearly as far below as it was originally above. Since, however, the tube was not a sufficiently good conductor to permit a rapid equalization of the potential, a potential difference of nearly $2\ V$ was momentarily established, and resulted in the rupture of a layer of the dielectric nearly twice as thick as that at de.

The oscillatory character of the electric discharge was first recognized by Henry, in 1842, as the explanation of the apparently anomalous magnetization produced in a needle by the discharge of a Leyden jar. Fedderson, some years later, found, by the use of Wheatstone's revolving mirror, that the spark actually consisted of a number of separate flashes. A method of calculating the period of the oscillation, which is very short, will be explained in Art. 458.

304. The Unit Jar. — If the inner and the outer coating of a Leyden jar are so arranged that a spark will pass whenever the potential difference reaches a certain definite value, the apparatus may be employed to measure in arbitrary units

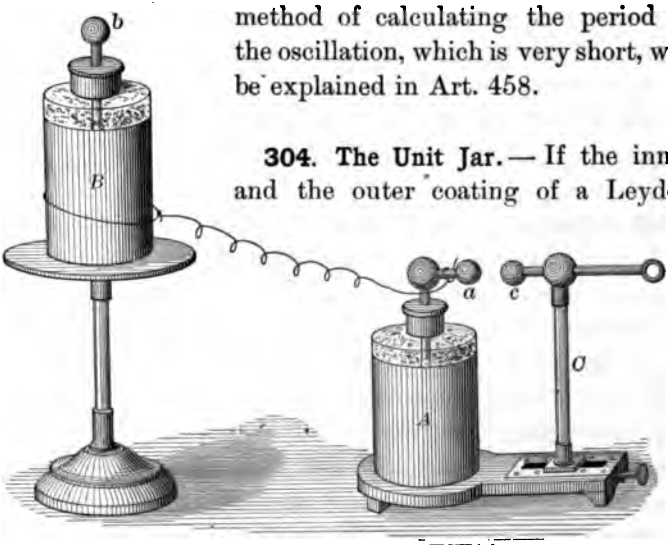


FIG. 222.

the quantity of electricity used to charge a condenser. Such an apparatus, known as *Lane's unit jar*, is shown in Fig. 222. The distinctive feature is the pillar *C* connected to the outer coat of *A* and the earth, and carrying a movable knob, *c*, by which the width of the spark gap may be adjusted. In order to measure the charge put into the jar *B*, let the latter be placed on an insulating stool and its outer coat connected with the inner coat of *A*. Then, if by means of an electrical machine a charge be given to the inner coat of *B*, an exactly equal charge will be induced on the outside of *B* and on the inside of *A*, and when the potential of *A* has been raised so as to strain the air between *a* and *c* to the breaking point, a spark will pass at *ac*, and *A* will be discharged, without, however, affecting the quantity of electricity which was introduced into *B*. Thus, by counting the sparks at *ac* during the process of charging, a definite knowledge of the quantity of electricity given to *B* is obtained.

305. Electricity is not Energy. — After having given the condenser *B* (Fig. 222) a charge equal to one of the arbitrary units just described, let it be discharged, and both the brightness and noise of the spark carefully noted. Then let the jar be charged with three units and again discharged. The spark will appear something like three times as long and very much brighter; the sound also will be greatly increased. Now, as the sound and heat which appear at the moment of discharge represent the energy which was spent in charging the jar, it follows that the energy of the second charge was more than three times as great as that of the first. But the quantity of electricity was only three times as great. Hence, it is evident that electricity cannot be energy, for they do not vary in the same ratio. That the

energy of a charged conductor is exactly proportional to the square of the charge may be shown as follows.

306. Energy of a Charged Conductor. — Suppose that an insulated conductor, A, is initially without charge and at a potential zero, and that a quantity of electricity, Q, is brought up to it in a succession of small charges, raising its potential to V. Now, as the initial value of the potential is zero, and the process of charging is a uniform one, and as the potential is proportional to the charge, it is obvious that the average value of the potential during the charging will be $\frac{1}{2}(V+0)$. But by definition of potential this is the average work done in bringing a unit quantity to A. Hence, the work done in bringing up Q units will be

(1) $$W = \tfrac{1}{2} VQ,$$

or, substituting the value of the potential in terms of the capacity,

(2) $$W = \tfrac{1}{2}\frac{Q^2}{C}.$$

Therefore, since C is a constant, the energy of a charged conductor varies as the square of the charge.

307. Atmospheric Electricity. — The resemblance of the electric spark to lightning was remarked by the first observers who succeeded in obtaining potential differences great enough to produce the disruptive discharge. The identity of the two phenomena was, however, first established by Franklin, who made the comparison of a number of points of resemblance, and proposed as a final test to draw electricity from a thunder cloud by means of a pointed rod erected upon an eminence.

Franklin himself succeeded in conducting the electricity from the clouds by a hempen string attached to a kite

that was put up at the approach of a thunder shower. He found, among other things, that sparks could be drawn from a key tied to the string on presenting the knuckle, and that a Leyden jar could be charged with the electricity so obtained.

The experiment with the pointed rod was actually tried by Dalibard, near Paris, about a month before Franklin performed his kite experiment; but the honor belongs to Franklin, for Dalibard obtained the idea from a paper of Franklin's which had been recently published.

308. Water-Dropping Collector.—The electrification of the atmosphere is by no means confined to the region of thunder clouds, but may be observed at all times by means of suitable apparatus. The instrument most commonly used for this purpose is Kelvin's water-dropping collector, shown in Fig. 223. A is a metal can placed on [illegible] the insulating stand B, and filled with water, which is allowed to discharge in a fine stream from the nozzle P, extending some distance beyond the wall.

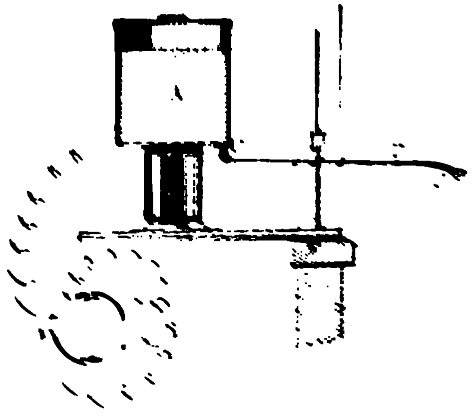


Fig. 223.

Suppose that the potential of the point P where the stream [illegible] [illegible] [illegible] the charge is [illegible].

that the water escapes in a fine stream breaking into drops at P. These drops will carry off the negative charge till the potential of the can has risen to that of the air at P. The measurement of this potential is usually effected by a quadrant electrometer, E, connected to the outside of the can. Observations made in the way described show that the potential of the atmosphere in fair weather is higher than that of the earth, but its value is constantly fluctuating in amount, as if produced by the passage of masses of positively charged air. In broken weather, and during rain, the potential is more often negative than positive, with abrupt changes of considerable magnitude, the sign often passing rapidly from minus to plus, or *vice versa.*

309. On the Cause of Atmospheric Electrification. — In the present state of science no adequate explanation of the occasion of atmospheric electrification can be given. It is possible that the friction of solid or liquid particles against each other may be one of the agents by which electrical separation is brought about. This conjecture is favored by the excessive electrification of the air observed during a snow-storm accompanied by wind.

Lenard discovered that air was electrified whenever drops of water were allowed to splash on the surface of a solid or on another body of water. That a certain portion of the electrification of the atmosphere may be traced to this cause is made probable by observations in the vicinity of waterfalls and along the seacoast, but no theory yet advanced satisfactorily accounts for such enormous differences of potentials as are necessary to produce sparks of a mile or more in length.

310. Return Shock. — If a conductor, forming a nearly closed circuit, be placed near a condenser, but without con-

tact, a spark may sometimes be observed to jump across the air gap of the conductor at the moment when the condenser is discharged. The explanation is that the oscillations of the condenser discharge produce a field of varying intensity, and may induce a charge on the conductor with sufficient potential difference to break down the dielectric in the short gap. This phenomenon will be more fully discussed in Chapter XXIX, but is mentioned here for the purpose of pointing out the fact that such induced discharge frequently occurs at the instant of the lightning flash, and a body is thus often said to be struck, although it may have been at a considerable distance from the path of the primary discharge.

CHAPTER XX.

MAGNETISM.

311. Natural Magnets. — Bodies which have the power of attracting to themselves small pieces of iron are called magnets. The name is derived from the province of Magnesia, where specimens of iron ore possessing this peculiar property were first obtained. This mineral, now known as magnetite, is widely distributed over the earth, but is not always found in the magnetic condition. When a natural magnet is dipped into iron filings, they will be observed to cling to the body in a very erratic manner; but if a hardened steel needle be first stroked with the magnet and then dipped into the filings, a large number will be found to cluster about the ends, leaving the middle portion bare. If such a magnetized needle be suspended, so as to move freely in a horizontal plane, it will ultimately come to rest in a direction nearly north and south. On account of this peculiar directive property, natural magnets have been called lode (*i.e.* leading) stones.

312. Definition of a Magnetic Pole. — A very long and slender uniformly magnetized needle behaves as if there were a center of force at each end, and all the rest of the needle were devoid of magnetism. The ends of such a magnet are called its poles. That end of the needle which would point north, if it were freely suspended, is called the north pole. Similarly, the south-seeking end is called the south pole. For simplicity of statement of the laws governing magnets, it is convenient to speak as if the ends of a

magnetized needle were charged with a definite quantity of a measurable physical magnitude called *magnetism*. It must, however, be distinctly understood that this is only a figure of speech, and that there is no reason for admitting magnetism to the rank of a physical entity. The fact actually observed is the action of one body on another through an intervening medium. One important advantage of assuming that the ends of a magnet possess a charge of magnetism is that many problems on the magnetic field become identical in form with those of the electrostatic field, which have been presented in the preceding articles. There is, however, this notable exception, that there is no conductor of magnetism, and in consequence it is impossible to magnetize a body entirely with north magnetism or south magnetism.

313. Magnetic Attractions and Repulsions. — If the north pole of a magnet be presented to a suspended magnetic needle, the north pole of the latter will be repelled but the south pole will be attracted. In the same manner, the south pole will be found to attract the north pole of the needle but to repel the south pole. These results may be reduced to the statement that like poles repel and unlike poles attract.

314. Law of Magnetic Force. — The law of action of one magnetic pole on another is precisely analogous to that given in Art. 271 for electric charges. Thus, let m be the strength of one pole, *i.e.* the quantity of magnetism at, say, the north end of an infinitely long thin rod magnetized uniformly, and m' the strength of the pole of another similar magnet; then, if the distance between these north poles be called r, the force of repulsion may be written

$$F = \frac{m\, m'}{\mu\, r^2}, \tag{1}$$

where μ is a constant, called the specific magnetic inductive capacity, or *permeability* of the medium. The law of the inverse square of the distance was first announced by Coulomb, in 1785, as the result of measurements on magnetized bars with the torsion balance, an instrument in which one magnet was suspended horizontally by a wire, and the force of repulsion between one of its poles and the pole of another magnet was measured by the torsion produced in the wire. This method does not afford a very accurate verification of the law. Another demonstration will be explained in Art. 320, by which the action of one magnet on another is calculated under the assumption that the force exerted by one pole on another varies inversely as the square of the distance. The truth of the assumption may be thus made to depend on observations which admit greater accuracy than those of Coulomb.

The dependence of the force upon the nature of the medium was first demonstrated by Faraday, who showed that the permeability μ in magnetic phenomena is the exact analogue of the specific inductive capacity K of dielectrics. The value of μ is taken arbitrarily as unity in empty space.

315. Definition of the Unit Pole. — The statement of the law of magnetic force,

$$F = \frac{\mathfrak{m}\,\mathfrak{m}'}{\mu\, r^2},$$

leads at once to the definition of the unit quantity of magnetism, or the unit pole. For making $\mathfrak{m} = \mathfrak{m}'$, and the other quantities units, the numerical value of $\mathfrak{m}$ becomes also unity. Accordingly, the unit pole may be defined as a pole which, when placed at the distance of one centimeter in a vacuum from an exactly similar pole, will repel it with the force of one dyne.

316. Magnetic Line of Force. — A magnetic line of force is a line which has everywhere the direction in which a small north pole would be urged.

If a system of lines of force be drawn through every point of a closed curve, the tubular surface so formed is called a tube of force, see Fig. 232.

317. Magnetic Field. — A magnetic field is any region through which magnetic forces act. Magnetic intensity at a point, or the strength of the field, is the force which would be exerted on a small magnetic pole placed at the point divided by the strength of the pole. If the pole $\mathfrak{m}$ exert the force F on the pole $\mathfrak{m}'$ at a distance, r, then the strength of the field $\mathbf{H}$ produced by $\mathfrak{m}$ is given by

$$\text{(2)} \qquad \mathbf{H} = \frac{F}{\mathfrak{m}'} = \frac{\mathfrak{m}}{\mu\, r^2}.$$

A magnetic field may be represented by drawing lines of force and equipotential surfaces in just the same manner as was done for the electric field.

It is evident that any small magnetized needle will, when brought into a magnetic field, set itself tangent to the line of force at the point. Thus, by carrying such a needle freely suspended through an unknown field, the region may be completely explored.

The lines of force run from positively magnetized surfaces to negatively magnetized ones. Magnetic lines of force are, however, subject to this limitation, that as many lines as leave one part of a body must return to some other part, or, in other words, a north pole is always associated with an equal south pole in the same body. The truth of this statement rests upon the experimental fact that a magnet, in a uniform field, experiences no force tending to move it as a whole. Thus, if a magnetic needle be placed in a small

vessel floating on the surface of water, the needle will orient itself under the influence of the earth's field, but the vessel will experience no motion of translation.

318. Magnetic Moment. — If a bar magnet be placed in a uniform field and at right angles to the lines of force, it will experience a couple depending on the strength of the field and a property of the bar known as its *magnetic moment*, which is defined as the quotient of the couple by the strength of the field. Thus, if **H** stand for the strength of the field, **M** for the magnetic moment, and **C** for the couple, under the conditions specified, then

$$\text{(3)} \qquad \mathbf{C} = \mathbf{HM}.$$

In an ideal magnet consisting of two poles at the extremities of a line, the magnetic moment is simply the product of the strength of one of the poles by the distance between them.

319. Intensity of Magnetization. — The quotient of the magnetic moment of a magnet by its volume is called the *intensity of magnetization.*

Thus, if the volume of the magnet be denoted by v, its magnetic moment by **M**, and the intensity of magnetization by **I**,

$$\text{(4)} \qquad \mathbf{I} = \frac{\mathbf{M}}{v}.$$

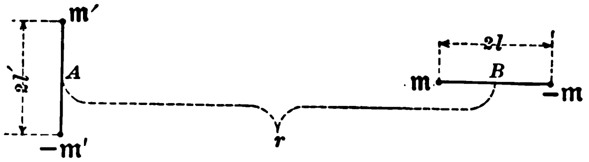


Fig. 224.

320. Action of One Magnet on Another. — Suppose that A (Fig. 224) is a small magnet whose length is $2\,l'$ and

whose pole strength is m′, placed at a distance, r, from another magnet, B, having a length, $2\,l$, and a pole strength, m. If r is large with respect to l, the force between m and m′ may be written

$$F_{\mathrm{m\,m'}} = \frac{\mathrm{m\,m'}}{\mu\,(r-l)^2}. \tag{5}$$

Then, since there is a similar force between m and $-$m′, the moment of the pole m on the magnet A is

$$M_{\mathrm{m}} = \frac{\mathrm{m\,m'}\,2\,l'}{\mu\,(r-l)^2}; \tag{6}$$

likewise the moment of $-$m in the opposite direction is

$$M_{-\mathrm{m}} = \frac{\mathrm{m\,m'}\,2\,l'}{\mu\,(r+l)^2}, \tag{7}$$

or, the total turning effect on A produced by B will be

$$M = 2\,\frac{\mathrm{m\,m'}}{\mu}\,l'\left\{\frac{1}{(r-l)^2} - \frac{1}{(r+l)^2}\right\}$$

$$= 2\,\frac{\mathrm{m\,m'}\,l'}{\mu r^2}\left\{\frac{1}{\left(1-\frac{l}{r}\right)^2} - \frac{1}{\left(1+\frac{l}{r}\right)^2}\right\}; \tag{8}$$

expanding the terms within the brackets by the binomial theorem and neglecting powers above the first,

$$M = 2\,\frac{\mathrm{m\,m'}\,l'}{\mu\,r^2}\left\{1 + \frac{2\,l}{r} - \left(1 - \frac{2\,l}{r}\right)\right\}$$

$$= 8\,\frac{\mathrm{m\,m'}\,l\,l'}{\mu\,r^3}. \tag{9}$$

It thus appears that the turning moment experienced by A varies inversely as the cube of the distance. If the magnets in Fig. 224 be so arranged that A is initially in the

plane of the meridian, the turning effect of B may be taken as proportional to the angular deflection of A, provided the angle is small.

Thus, the truth of the assumption that the force between two magnetic poles varies inversely as the square of the distance may be verified.

321. Terrestrial Magnetism. — The ancients ascribed the orientation of a freely suspended magnetic needle to some mysterious influence of the pole-star, whence the term *pole* came to be used in connection with the ends of the needle.

About 1600 Gilbert made the discovery that the earth itself possesses properties comparable to those of a magnet. The complete specification of the earth's magnetic field at any point requires a knowledge of three quantities, called, respectively, the declination, the dip, and the intensity.

322. Declination. — The direction of a suspended needle varies greatly with the place on the earth at which the observation is made. In order to specify the position of the needle in a horizontal plane, the angle between the vertical plane through the needle and the plane of the geographical meridian is called the *magnetic declination* at the place considered. This angle is readily obtained by observation of the mean position of a swinging magnetic needle, suspended by a fiber without torsion.

323. Dip. — The position assumed by a magnetic needle, supported at its center of gravity and free to turn about a horizontal axis, also varies with its position on the earth's surface. The minimum angle made by such a needle with the horizontal plane is called the *inclination*, or *dip*. In the northern hemisphere it is the north pole which is depressed; in the southern hemisphere it is the south pole.

The total intensity of the earth's field at a point is usually divided into a vertical and a horizontal component. If the total intensity be denoted by T, the horizontal component by H, the vertical component by V, and the dip by δ,

$$H = T \cos \delta. \quad (10)$$
$$V = T \sin \delta.$$

MAGNETIC CONSTANTS FOR A.D. 1900.

Locality.	Declination.	Dip.	Total Intensity, C. G. S. Units.
London	16° 16′ W.	67° 9′ N.	0.47
St. Petersburg . .	0° 30′ E.	70° 46′ N.	0.48
Berlin	9° 30′ W.	66° 43′ N.	0.48
Paris	14° 30′ W.	64° 55′ N.	0.47
Rome	10° 0′ W.	58° 0′ N.	0.45
New York . . .	9° 12′ W.	70° 6′ N.	0.61
Washington . . .	4° 35′ W.	70° 18′ N.	0.60
San Francisco . .	16° 42′ E.	62° 20′ N.	0.54
Mexico	8° 0′ E.	45° 1′ N.	0.48
Cape Town . . .	29° 24′ W.	58° 2′ S.	0.36
Sydney	9° 36′ E.	62° 45′ S.	0.57
Bombay	0° 36′ E.	20° 38′ N.	0.37
Tokio	4° 6′ W.	49° 52′ N.	0.45

The dip is obtained from observations made on a needle delicately suspended from a horizontal axis, and placed so as to swing in the magnetic meridian. The horizontal component of the earth's magnetism may be obtained from observations on the period of vibration of a magnet suspended horizontally. The expression for the period of vibration of such a magnet is similar to that for a compound pendulum, namely,

$$P^2 = \frac{4\pi^2 \Sigma mr^2}{MH}, \quad (11)$$

where H is the horizontal component of the earth's field, **M** the magnetic moment of the magnet, and Σmr^2 the moment of inertia of the bar.

Values of the magnetic constants in different localities, for the year 1900, are shown in the preceding table.

324. Magnetic Charts. — A knowledge of the magnetic condition at every point of the earth's surface is obviously of the highest importance to geodesy and navigation. The record of observations at different points is usually made by drawing a system of lines on a map.

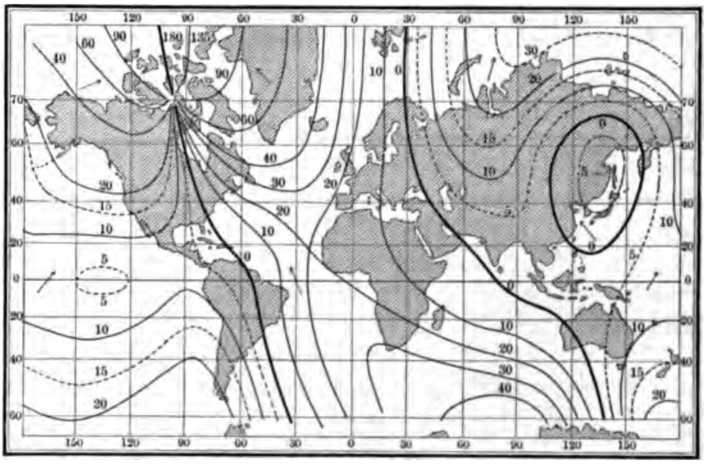


FIG. 225.

In Fig. 225 these lines are so drawn that any one of them passes through all points having the same declination. Such a line is known as an *isogonic*, or line of equal declination.

In Fig. 226 each line of the system passes through all points which have the same dip. Such a line is called an *isoclinic*, or line of equal dip.

In Fig. 227 lines are drawn so as to pass through all

points having the same horizontal intensity. Such lines are known as *isodynamics*.

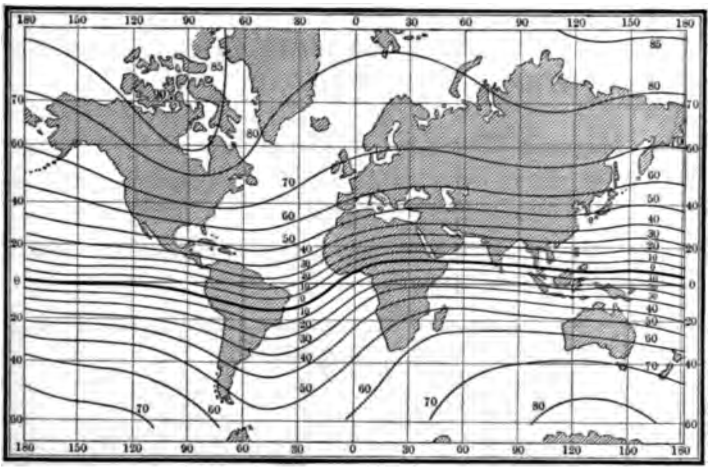


FIG. 226.

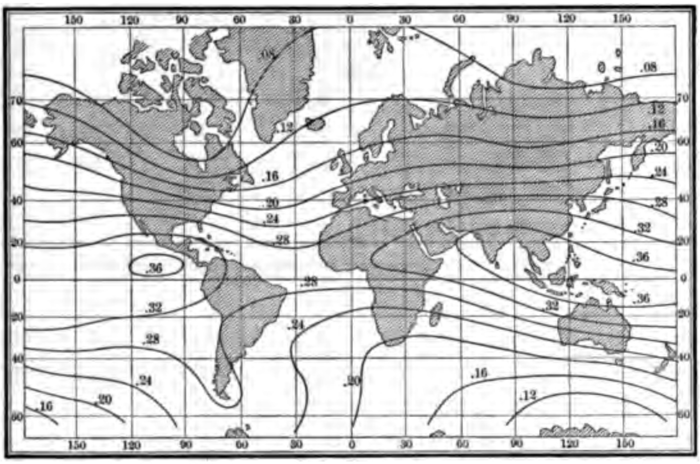


FIG. 227.

Still another system is shown in Fig. 228, in which the approximately north and south lines are drawn so that they

have at every point the direction of the horizontal component of the earth's field. Such lines are known as *magnetic meridians.* It will be observed that they converge to points in each hemisphere known as the magnetic poles. The one

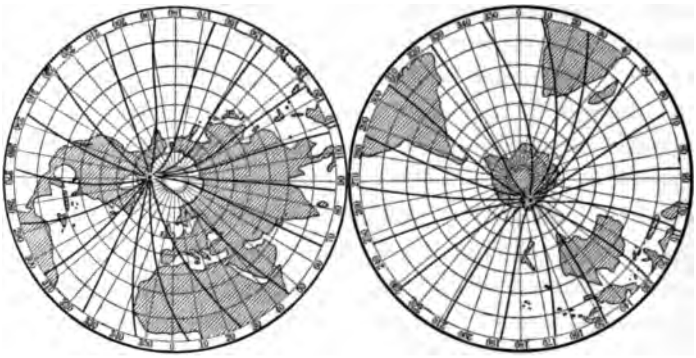


FIG. 228.

in the northern hemisphere is located 70.5° N., 97° W. The south magnetic pole has never been reached, but it is believed to be not far from 73° S., 150° E. It is evident that they do not lie at the extremity of a diameter.

325. Biot's Hypothesis. — Biot has suggested that the magnetic field of the earth could be roughly imitated by placing two magnetic poles within the earth, as at N and S (Fig. 229), situated at a distance apart small compared to the earth's radius, on a line inclined 20° to the geographical axis PP'. The angle made by any line of force in the figure with the earth's surface at the point of intersection is that which has been defined as the dip. Biot's hypothesis is equivalent to the supposition that the earth is a uniformly

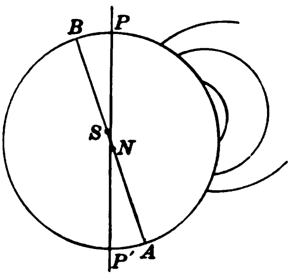


FIG. 229.

magnetized sphere. The actual phenomena of terrestrial magnetism are, however, much too complicated to admit of explanation by this simple assumption.

326. Variations of the Earth's Field. — The magnetic forces of the earth are subject to a variety of changes, which may for convenience be grouped as follows :

1°. Solar variations, or those changes which depend on the position of the sun. The diurnal variation in declination is most easily observed. The western deviation shows a maximum about 1 P.M., and a minimum at 10 P.M.; but the total daily change is rarely more than 10′ of arc. There is also an annual change in the intensity, which reaches a maximum about June and a minimum about February, in the northern hemisphere.

2°. Lunar variations, or certain slight changes which depend on the position of the moon.

3°. An eleven-year period, or a change in declination, which reaches a maximum apparently at the epoch of the maximum number of spots on the sun.

4°. Secular variation, or a continuous change of long period, which has been going on ever since magnetic observations began to be made in A.D. 1580. The period of the secular variation in declination may approximate 340 years. The change near the agonic line in the United States in 1890 was 3′ westerly increase. At Greenwich the change was about 7′ westerly decrease.

5°. Irregular disturbances, which affect the magnetic elements simultaneously over the whole earth. These sudden variations are sometimes called magnetic storms, and frequently coincide with brilliant auroral displays. Nothing is at present known of the causes of these disturbances, but it seems probable that they are exterior to the body of the earth.

327. Mariner's Compass. — A compass needle on shipboard is subject to serious disturbance from the motion of the ship and from the presence of considerable masses of iron. Steadiness in a heavy sea is obtained by making the vibration period of the needle very long. Fig. 230 shows the disposition of parts in the compass designed by Lord Kelvin.

Eight small needles of thin steel wire from 2 in. to $3\frac{1}{2}$ in. long are fastened to two parallel silk threads, and suspended from a light circular aluminum rim 10 in. in diameter. The rim is connected by radial silk threads with an aluminum ring at the center, which rests on a cup containing a sapphire crown. This jewel is itself supported on an iridium point, about which it turns freely, carried by a pillar attached to the bottom of the compass bowl. The aluminum rim with the silk threads forms a platform for a light paper circle on which are engraved the points of the compass, and a circle graduated to degrees. The weight of the whole movable piece, or "card," as it is usually called, is about 170 grains (11 grams), and its period about 42 sec.

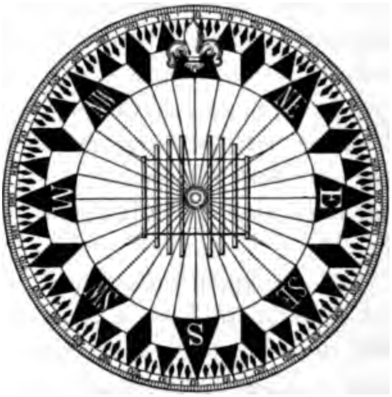


FIG. 230.

To keep the card horizontal and shield it from tremors in the ship, the compass bowl is carried by a gimbal ring supported on knife edges resting on stirrups which are hung by chains from an elastic wire ring of rope pattern. This ring has two sockets fixed at the end of a diameter, which rest on two balls attached to the rim of the binnacle stand. The deviation of the compass produced by the magnetization of

the iron in the cargo, the ship or her fittings, cannot be permanently corrected on account of the erratic variations in the ship's magnetism. That portion which is approximately permanent may be counteracted by magnets placed in the neighborhood of the compass; likewise, that part which is transient, *i.e.* dependent on the actual strength of the earth's field, may be compensated by pieces of soft iron properly disposed. There always remains, however, a considerable amount of magnetism, which strictly belongs in neither class, but varies in an entirely lawless way. This can be taken account of only by an occasional determination of the errors of the compass on several courses, a process known as "swinging the ship," and a readjustment of the correctors.

328. Magnetic Induction.—When a piece of iron is brought into a magnetic field, it assumes a magnetic condition, and is said to be magnetized by induction. Thus, if a piece of soft iron, originally devoid of magnetism, be placed near some iron filings, and then approached by a strong magnet, the filings will be attracted to the ends of the iron as if it were itself a magnet, as long as it is in the magnetic field, but as soon as the inducing body is withdrawn, the magnetic effects will nearly or quite disappear. Assuming for the present that the magnetism of the iron is wholly induced, the mathematical solution of this case is precisely analogous to that of a dielectric body placed in an electric field. If an iron sphere be placed in a uniform field in air, the arrangement of the lines of force, originally horizontal, will be that represented in Fig. 231.

As iron is a substance whose specific magnetic inductive capacity is greater than that of air, the equipotential surfaces will be crowded together so as to produce a more rapid variation of the potential at the sides than would be the case

in the undisturbed field. The body thus appears to be magnetized negatively where the lines of force enter it, and positively where they leave it.

If the permeability of the sphere is less than that of the surrounding medium, the distortion of a uniform field by the presence of the body will be that of Fig. 231 turned through a right angle, the former lines of force being now regarded as lines of the equipotential surfaces, and the former lines of the equipotential surfaces as lines of force. The shifting and crowding of the equipotential surfaces into the body has produced a change in the potential on each side, as if the sphere were magnetized positively where the lines of force enter it, and negatively where they leave it.

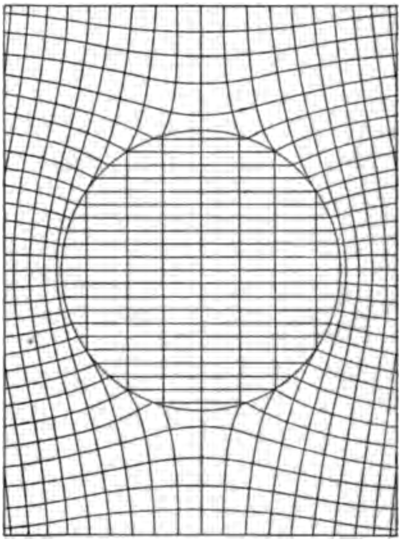


FIG. 231.

The description of the behavior of a dielectric body in an electric field, as given in Art. 283, applies equally to bodies in a magnetic field, if the word permeability be substituted for specific inductive capacity. The different cases may be enumerated as follows:

Case 1. μ within greater than μ without.

a. An elongated body in a uniform field will set with its axis parallel to the lines of force.

b. A spherical body in a non-uniform field will be urged toward the part where the intensity is numerically greatest.

c. An elongated body in a non-uniform field will set either parallel to or across the lines of force according as the effect described in *a* or in *b* predominates.

Case 2. μ within less than μ without.

d. An elongated body in a uniform field will set with its axis parallel to the lines of force.

e. A spherical body in a non-uniform field will be urged toward that part of the field where the intensity is numerically least.

f. An elongated body in a non-uniform field will set with its axis either parallel to or across the lines of force according as the effect described in *d* or in *e* predominates.

The values of the permeability for different bodies were first investigated by Faraday, who gave the name *paramagnetic* to those bodies in which μ is greater than for empty space. Iron, nickel, and cobalt are the only substances in which the permeability is notably greater than unity, and hence have well-marked magnetic properties. In the case of bismuth, antimony, and copper the value of μ was found to be a little less than one.

Since elongated masses of these bodies, in very intense non-uniform fields, set themselves across the lines of force, Faraday called them *diamagnetic*. The effect described in paragraph *d* is so small in these bodies that it has hitherto escaped detection. The numerical values of the permeability are further discussed in Arts. 332 and 338.

329. Hydrokinetic Analogy of Magnetic Induction. — The laws of magnetic induction may be most simply stated, without committing one's self to any physical theory of magnetism, in terms of the flux of an incompressible fluid through tubes of force. Thus, suppose that *ab* (Fig. 232) is a tube of force filled with such a fluid in steady motion, and that to maintain

this motion requires the application at each point of a force proportional to the velocity at that point. Then, because of the incompressible nature of the fluid, it is evident that the same quantity measured in units of volume must pass any cross section of the tube in the unit time, or, what amounts to the same thing, the velocity at any point will be inversely as the cross section. Suppose that in a tube of uniform cross section, a, a stream whose length was l passed in the time t. The expression for the velocity may then obviously be written

$$v = \frac{l}{t} = \frac{al}{at} = \frac{\text{volume}}{\text{area} \times \text{time}}.$$

This latter term, though numerically the same as the velocity, may be more easily measured experimentally. It is of the nature of a flux which may be defined as the quantity of a fluid which passes any section per unit area, per unit time.

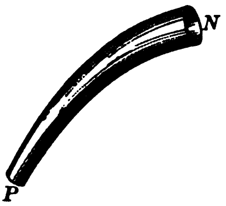


FIG. 232.

If a tube of flow pass from one medium to another, it suffers a sort of refraction, which may be thus determined: Let F_1 be the force at any point in one medium, and of such nature that it may be calculated from the rate of change of potential. Also, let v_1 be the flux produced by F_1, and k_1 a constant of the medium which determines the numerical relation of v_1 to F_1.

Then, since by hypothesis the flux is proportional to the force,

(12) $$v_1 = k_1 F_1.$$

Likewise for the second medium,

(13) $$v_2 = k_2 F_2.$$

Let XY (Fig. 233) be the interface between the media k_1 and k_2. The component of the force parallel to this interface will be the same in both media. For, let a, b be two points in k_1, and c, d two in k_2. Since the force by assumption is derived from a potential, the force at a will be the rate of change of potential in going from a to b, and, similarly, the force at c is the rate of change of potential in passing from c to d. If, now, a and b are made to coincide, respectively, with c and d at the interface, the change of potential from a to b becomes identical with that from c to d. Therefore, if i_1 be the angle between the tube and the normal in the first medium, and i_2 the corresponding angle in the second medium,

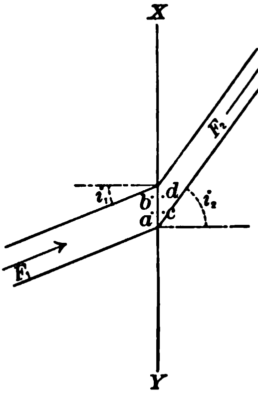


Fig. 233.

(14) $$F_1 \sin i_1 = F_2 \sin i_2.$$

Again, let a small tube be drawn crossing the interface, and suppose that one end, having the area s, is in k_1, as at ab, and the other end, with the same area, is just within k_2, at cd. Then, since the fluid is incompressible, the total flow through each end of this tube must be the same, or,

(15) $$v_1 \cos i_1 \cdot s = v_2 \cos i_2 \cdot s.$$

Substituting the values from equations 12 and 13,

(16) $$k_1 F_1 \cos i_1 = k_2 F_2 \cos i_2;$$

whence, dividing,

(17) $$\frac{1}{k_1} \tan i_1 = \frac{1}{k_2} \tan i_2;$$

that is, a tube, in passing from a medium having a less value of k to one having a greater, is bent away from the normal.

Equation 15 shows that the component of the flux normal to the surface of separation is constant.

If, now, in this hydrokinetic model the quantity called the force be replaced by the intensity of the field $\mathbf{H}$, and the constant k by μ, then the analogue of the flux v is called the *magnetic induction.*

Denoting the induction by $\mathbf{B}$, the equation connecting these three quantities is

$$\mathbf{B} = \mu\mathbf{H}. \tag{18}$$

330. Kelvin's Definition of Induction and Intensity. — Lord Kelvin has given the following definitions of magnetic intensity and induction. Suppose that a substance capable of magnetization be placed in a magnetic field, and that a small needle-shaped cavity, with its axis parallel to the direction of magnetization $\mathbf{I}$, be made in the body.

Then the force $\mathbf{H}$, which would be experienced by a unit pole placed at the center of this cavity, is called the *magnetic intensity* at that point in the magnetizable medium. If, on the other hand, the unit pole be placed in a disc-shaped cavity cut at right angles to the magnetization, the force which it would experience is the induction $\mathbf{B}$, at that point. It may be shown that $\mathbf{B}$ can be expressed in terms of $\mathbf{H}$ and $\mathbf{I}$, by the equation

$$\mathbf{B} = \mathbf{H} + 4\pi\mathbf{I}. \tag{19}$$

331. Magnetic Susceptibility. — When a body which is capable of magnetic induction is brought into a magnetic field, the dependence of the induced magnetization $\mathbf{I}$ upon the intensity of the field $\mathbf{H}$ is usually expressed by the relation

$$\mathbf{I} = \kappa\mathbf{H}, \tag{20}$$

where κ is the measure of a physical property of the body called its *susceptibility*.

By substitution in equation 19,

$$\mathrm{B} = (1 + 4\pi\kappa)\,\mathrm{H}, \text{ or,} \tag{21}$$

$$\mu = 1 + 4\pi\kappa. \tag{22}$$

332. Discussion of the Magnetic Constants of Iron. — The magnetic properties of iron have been found to depend in an important manner upon the nature of each specimen, its previous history, its temperature, and the strength of the field in which it is placed. The general character of the changes which occur in the magnetic constants of wrought iron as the strength of the field is increased is shown in the following table.

Magnetizing Force.	Magnetization.	Susceptibility.	Induction.	Permeability.
H	I	κ	B	μ
0.3	3	10	41	128
1.4	32	23	413	299
2.2	117	53	1460	670
3.5	574	164	7230	2070
4.9	917	187	11540	2350
6.7	1078	161	13520	2020
10.2	1173	115	14840	1450
22.3	1249	56	15710	705
78	1337	17	16900	215
208	1452	7	18500	89
585	1530	2.6	19800	34
24500	1660	0.067	45300	1.9

The record of observations of the properties of any specimen may be conveniently studied by the aid of such a

diagram as Fig. 234, in which the magnetizing forces are plotted as abscissas and the values of the induction as ordinates.

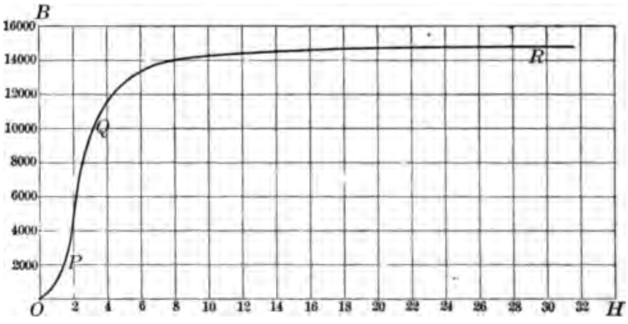


Fig. 234.

Although the form of the curve varies considerably with different specimens, there may usually be distinguished three stages in the process of magnetization. While the magnetizing force is small, as from O to P, the induction increases more rapidly than the magnetizing force. From P to Q the ratio of **B** to **H** is nearly constant, but from the region of Q this ratio continuously decreases, and the curve finally approaches a horizontal straight line. The condition in which the magnetization no longer increases with the strength of the field is called the state of ***saturation.*** In the figure the iron is near saturation when

$$\mathbf{H} = 14 \text{ C.G.S. units.}$$

The physical interpretation of these three stages of magnetization is discussed in Art. 336.

333. Paramagnetic Bodies in Intense Fields. — The accompanying diagram, in which the coördinates are μ and **B**, exhibits the magnetic properties of cobalt, nickel, steel, and

cast iron, as compared with wrought iron. In the specimen of manganese steel tested, the permeability was sensibly con-

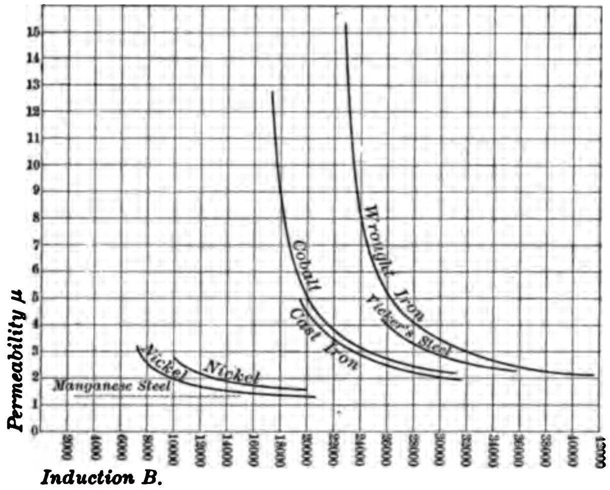


FIG. 235.

stant at about 1.4. All the curves show a saturation limit. The values of **I** and **H**, when saturation sets in, are shown in the following table.

TABLE OF SATURATION VALUES OF **I** AND **H**.

	I	H
Wrought Iron	1700	2000
Vicker's Steel	1600	15000
Cobalt	1300	9000
Cast Iron	1200	4000
Hard Nickel	400	8000
Annealed Nickel	515	7000
Manganese Steel	200	7000

Manganese, chromium, oxygen, and a number of other substances are feebly paramagnetic.

334. Temperature Effects. — The variation of the temperature of iron is accompanied by peculiar and often considerable changes in its magnetic properties. The results of Hopkinson's investigation on the permeability of soft iron in a weak field, as the temperature was raised, is shown in Fig. 236.

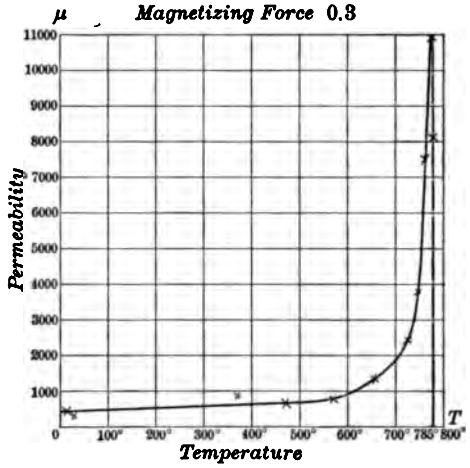


Fig. 236.

The permeability is seen to change but little while the temperature increases from 0° to 600° C. It then rises with great rapidity till a value of 11,000 is reached at 775°, after which it falls immediately to 1.1 at 786°. In stronger fields there is no such striking increase of the permeability, but there still remains the characteristic, though more gradual, disappearance of its magnetic properties in the region of 785°. The temperature at which a body becomes non-magnetic has been called by Hopkinson the *critical temperature.* Other phenomena have been observed in iron which indicate that the physical constitution of this metal undergoes some notable change of constitution at this point which varies from 690° to 870° C. in different specimens. Thus, Barrett has shown that if a piece of steel be heated to bright redness and allowed to cool rapidly, there is a sudden check in the process when this temperature is reached. The dull red surface of the body momentarily increases in brightness

and then goes on diminishing. This phenomenon, known as *recalescence*, is accompanied by a slight expansion. It has been observed that the electrical resistance and thermo-electric properties of iron also undergo peculiar alterations in the same region.

335. Magnetism a Molecular Phenomenon. — In the previous discussions, for convenience of mathematical treatment, bodies have been considered as possessing magnetic charges analogous to electric charges, but a number of the experimental results obtained, *e.g.* the phenomenon of saturation, the loss of magnetism at the critical temperature, and the absence of magnetic conductivity, show that there is some essential distinction between magnetization and electrification. The peculiarities named have their explanation in the fact that magnetization is a molecular phenomenon.

When a magnet is divided into two or more parts, each part is found to be a complete magnet, so that if this subdivision could be continued until the parts were reduced to molecular dimensions, there is no reason to doubt that these ultimate particles would themselves be magnets. This conclusion is corroborated by the behavior of an iron bar when tapped in a magnetic field. Thus, if a soft iron bar, AB, of considerable length, be presented to a magnetic needle while held in the direction of the earth's lines of force, the lower end will be found to have become a north pole by induction, whether the end B or A be the lower. But if the bar be tapped with a hammer while still parallel to the lines of force, it will be observed that the magnetization of the iron has been greatly increased and given a certain permanence, for on reversing the bar, the lower end will now be found to be a south pole.

These observations are most satisfactorily accounted for

on the hypothesis that each molecule is a permanent magnet, but under a certain constraint as to its orientation. On application of a small magnetizing force, there is a partial turning in the direction of the field, but on removal from the field the molecules return to their original positions. If, however, the molecules be jarred when in a magnetic field, a much larger number than before will be able to arrange themselves in the direction of the field, so that the induced magnetization is increased by the process. On removing the magnetizing force, the molecules are probably unable to regain their original positions of equilibrium, and a certain permanent magnetization results.

336. Three Stages of Magnetization. — Ewing has shown that the constraint to which the molecules of iron appear to be subject may be satisfactorily accounted for by their mutual action as magnets.

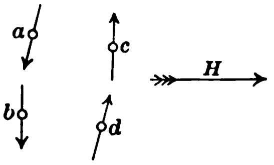


FIG. 237.

Let Fig. 237 represent a group of four molecules, *a*, *b*, *c*, *d*, placed at the corners of a square, and let the direction of their magnetic axes be indicated by the arrows.

The position represented is obviously one which the system might assume under the mutual action of the magnets. If, now, a small force be supposed to act in the direction of **H**, each magnet would be deflected, but on the disappearance of the directive force, would return to its original position. This action corresponds to the portion *OP* of the curve of induction (Fig. 234).

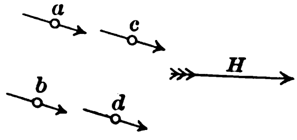


FIG. 238.

If the value of **H** is steadily increased, the arrangement of Fig. 237 becomes unstable, and the magnets pair off in a new grouping, such as Fig. 238. This change corresponds to the second stage of the magnetization, represented by *PQ* (Fig. 234). If **H** be increased still more, the group will be compelled to take the arrangement of Fig. 239, which is evidently the state of saturation. The change from Fig. 238 to Fig. 239 corresponds to the part of the curve *QR* (Fig. 234). If the directive force be withdrawn, the magnets will return to the condition of Fig. 238, which is different from the original grouping, and hence represents a residual magnetization.

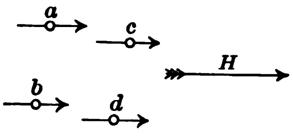


FIG. 239.

337. Hysteresis. — The residual effects observed in the study of the magnetic properties of iron may be represented on the **BH** diagram by plotting the values of the induction as the field is gradually weakened. The curve *RC* (Fig. 240) so obtained is found to differ considerably from *OR*, the curve of magnetization.

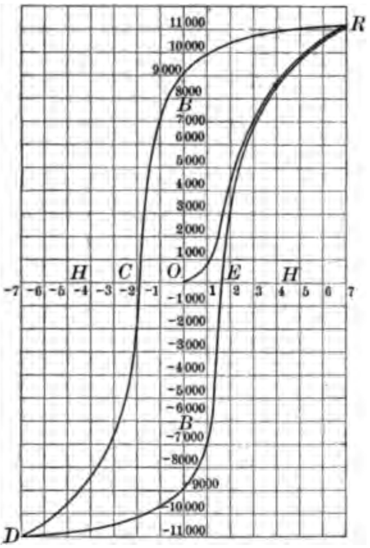


FIG. 240.

It appears, in this specimen of iron, that to bring the value of **B** to zero requires the application of a force, *OC*, in the negative direction. The value of the reversed magnetic force necessary to bring the curve to the **H** axis is

called the *coercive force.* In hard steel it is something like 100 C. G. S. units, in soft steel it is about 20 units, and in soft iron 2 units or less. If the reversed force be still further increased, the curve takes a form CD. If at the point D the field be again reversed and increased, the curve returns to R by the path DER, having enclosed a certain area during the cycle, so that there are always two values of $\mathbf{B}$ for every value of $\mathbf{H}$, and a similar result is obtained in every case in which $\mathbf{H}$ is varied so as to return to its original value.

Since the value of $\mathbf{B}$ with increasing field is always less than for the same value of $\mathbf{H}$ in a decreasing field, the phenomenon has received the name *hysteresis*, *i.e.* a lagging behind.

The area enclosed by any loop in the **BH** curve may be shown to represent the energy dissipated in any cycle of operations. A knowledge of this quantity is of great importance in the design of electromagnetic machinery.

338. Diamagnetic Substances. — It was pointed out in Art. 328 that, when a body is placed in a medium having a greater permeability than itself, it exhibits certain properties which were termed by Faraday diamagnetic.

The name diamagnetic substance is restricted to those bodies having a permeability less than that of vacuous space, for which μ is taken arbitrarily as unit.

Bismuth, the most highly diamagnetic substance known, has a value of μ about 0.9998. The magnetic forces developed in bismuth are thus almost infinitesimal compared to those in iron. Other diamagnetic substances are antimony, zinc, mercury, lead, silver, copper, gold, water, alcohol, and sulphur.

EXAMPLES.

1. If the strength of a magnetic north pole be 85 units, what would be the magnetic intensity at a distance of 15 cm. in air, due to this pole alone, and what force would a south pole of 35 units experience at this point?
Ans. $H = 0.38$ gm.$^{\frac{1}{2}}$ / cm.$^{\frac{1}{2}}$ sec.
$F = 13.2$ dynes.

2. If a pole of 280 units exerts a force of 0.93 gram weight on another pole at a distance of 3.2 cm. from it in air, what is the strength of the second pole? *Ans.* $\mathfrak{m} = 3.33\,(10)^3$ gm.$^{\frac{1}{2}}$ cm.$^{\frac{3}{2}}$ / sec.

3. What is the strength of the magnetic pole which experiences a force of 12.7 dynes, in a field whose intensity is 0.179 unit?
Ans. 71 gm.$^{\frac{1}{2}}$ cm.$^{\frac{3}{2}}$ / sec.

4. If a north pole of 78 units be placed at one corner of an equilateral triangle measuring 15 cm. on a side, and an equal south pole at another corner, what would be the intensity and direction of the field at the third corner?
Ans. 0.346 gm.$^{\frac{1}{2}}$ / cm.$^{\frac{1}{2}}$ sec.
parallel to the line *NS*.

5. A magnetic needle is suspended at some distance above and parallel to a horizontal bar magnet, which lies in the magnetic meridian. When the north end of the bar points northward, the period of the needle is 7.43 sec., but when the magnet is reversed, the period of the needle is 4.31 sec. What would be the period of the needle in the earth's field alone? *Ans.* 5.26 sec.

6. A weight of 1 gm. placed on the upper end of a dipping needle reduced the inclination from 65.2° to 39.5°. What weight would be necessary to bring it to the horizontal? *Ans.* 1.62 gms.

7. A horizontal magnet is suspended freely by a wire in the magnetic meridian. On twisting the upper end of the wire through 75° the magnet is deflected through 23.4°. What additional rotation of the top of the wire will be necessary to turn the magnet at right angles to the meridian? *Ans.* 144°.

8. If a magnet vibrate in a period of 4.98 sec., where the horizontal intensity is 0.182 unit, what will be its period where the intensity is 0.251 unit? *Ans.* 4.24 sec.

CHAPTER XXI.

THE ELECTRIC CURRENT.

339. Contact Difference of Potential or Volta Effect. — When two dissimilar bodies are brought in contact and afterwards separated, they are, in general, found to be oppositely electrified.

For instance, let C (Fig. 241) be a plate of copper connected to the earth, and Z a plate of zinc attached to an electroscope, E, and so arranged that after it has been put in communication with C it may be removed from the vicinity of the latter. Also, suppose that the whole experiment is carried on in air. If Z be momentarily touched to C and then removed, the gold leaf G will be observed to move toward N, the negatively charged pole of the electroscope, indicating that the potential of Z rises as it is removed; that is to say, Z is positively charged with respect to C, the difference of potential reaching a value not far from one volt, when the plates are separated a great distance.

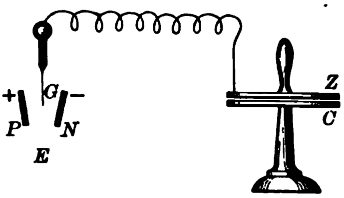


FIG. 241.

This potential difference represents a difference of chemical attraction in the plates for the oxygen of the air, and may be calculated from the heat of combination of oxygen with zinc and copper, respectively. If the plates had been immersed in some other gas, as *e.g.* H_2S, the potential difference on separating the plates would have been entirely different.

The potential difference which is observed on bringing a glass rod in contact with silk (Art. 264) has the same origin as that just described, though since the bodies are non-conductors the electrification will occur only at the points of contact. However, by the operation of rubbing, fresh points are brought in contact, and the small charges are piled up till the whole surface is electrified.

340. Explanation of Contact Potential Difference. — If the molecule of oxygen be regarded as the union of a positively charged atom with a similar negatively charged atom, then the contact difference of potential, described in the preceding article, may be understood as arising in the following manner.

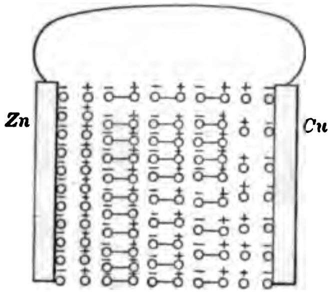


Fig. 242.

Let *Zn* and *Cu* (Fig. 242) be plates of zinc and copper joined by a copper wire and surrounded by an atmosphere of oxygen. The plates will then be at the same potential, and if the temperature is the same throughout it may be assumed that there is no electromotive force at the points of contact between the wire and the plates.

At the surfaces of the zinc and copper plates there will occur a union between the metals and the negatively charged oxygen atoms, and a consequent liberation of the positive atoms; and since zinc has a greater affinity for oxygen than copper, there will be more free positive atoms near the zinc than the copper plate. In this way there is produced a difference of potential between each plate and the layer of charged atoms next to it.

These amounts, as calculated from the heats of combination with oxygen, show that the layer of atoms next the zinc should be about 1.8 volts above the zinc, and that the layer next the copper should be about 0.8 volt above this plate. Accordingly, since the metals are at the same potential, there will be a fall of potential of one volt in passing from the layer of atoms next the zinc through the gas to the layer next the copper. The intermediate molecules will arrange themselves along the lines of force, the positive atoms experiencing a stress in the direction of the copper plate and the negative atoms one in the direction of the zinc. If the bond uniting any two atoms of the gas is strong enough to support the stress thus imposed, the whole system will remain, electrically, in virtual equilibrium, and such is the case with air when the plates are separated by any considerable distance. A gas thus behaves as a non-conductor as long as the intensity of the field is not sufficient to rupture the bond uniting the parts of the molecule. If, however, this bond is not sufficiently strong to sustain the stress put upon it in the field between the plates, the atoms will pair off in new ways, so that on the whole there will be a procession of positively charged atoms toward the copper and negatively charged ones toward the zinc through the gas, while in the wire there will be a flow of electricity from the copper to the zinc.

Something of this sort has been observed when the metal surfaces were very clean and brought very close together.

341. Voltaic Cell. — If the plates of Fig. 242 be immersed in an electrolyte, that is, a compound liquid which will not sustain a potential difference exceeding a certain small amount without chemical decomposition, there will be a continuous flow of electricity through the wire. Thus, sup-

pose that the two metals are connected by sulphuric acid, the molecule of which is regarded as consisting of the positive group, or radical H_2, and the negative radical SO_4. Then, if a potential difference is momentarily established, as in Fig. 242, between layers of the liquid next to the plates, the bond uniting the hydrogen to the acid radical SO_4 will be ruptured, the hydrogen going toward the copper plate, where it collects in minute bubbles. At the same time the sulphion works its way toward the zinc, and uniting with it forms $ZnSO_4$. The copper thus receives a positive charge from the hydrogen, and the zinc a negative one from the sulphion; but as they are connected by a conducting wire, a continuous discharge occurs, which, according to the ordinary convention, is known as a current from the copper to the zinc. If, however, the external circuit is broken, the copper end will be found at a certain constant difference of potential above the zinc end. The arrangement just described is known as the *voltaic cell.* Its essential elements are two dissimilar conductors, and an electrolyte capable of a reaction with one of the conductors.

Any part of an apparatus at which it is assumed that the current enters is called the *anode*, *i.e.* the way in (up); and the part by which the current leaves is called the *kathode*, *i.e.* the way out (down). Collectively these parts are termed electrodes. In the cell just described the zinc is the anode and the copper the kathode.

342. History of the Voltaic Cell. — Previous to 1786 no electrical phenomena were known other than those which were produced by the charges obtained from the electrical machine. In the year mentioned, Galvani, having observed that some freshly prepared frogs' legs, lying near an electrical machine, suffered a convulsive twitch at the moment of

discharge, sought to test the influence of atmospheric electricity by suspending some similar preparations on a copper hook from the iron railing of a balcony. This experiment yielded only a negative result, but on attempting to disconnect the hook, Galvani noticed that a specimen which came in contact with the railing experienced the same convulsion which he had before observed at the instant of an electrical discharge. The experiment attracted wide attention. Galvani himself attributed the phenomenon to an electric charge generated in the muscles of the leg. This explanation, however, was rejected by Volta, a professor in the University of Padua, who maintained that the true cause was the contact of dissimilar metals. In support of his view he devised an experiment equivalent to that of Art. 339. Then, pursuing the same line of proof, he tried arranging a series of discs of zinc, silver, and moistened cloth, after the manner shown in Fig. 243. On connecting the top and bottom discs of such a "pile" with the wetted fingers, a perceptible shock was obtained. Volta further showed that a number of compound

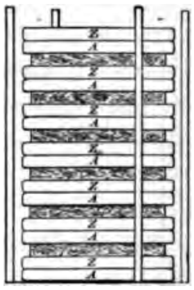


FIG. 243.

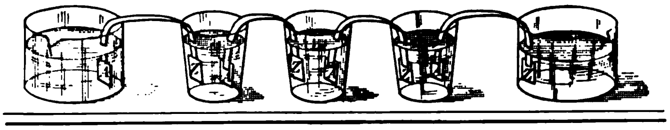


FIG. 244.

metal strips, half silver and half zinc (Fig. 244), bent so as to dip into a series of cups containing dilute acid, was an arrangement electrically equivalent to his "pile."

As the jars were usually set in a circle, so as to bring the extremities nearer together, it received the name *Crown of Cups*.

343. Local Action. — Specimens of commercial zinc usually contain impurities, such as bits of iron, carbon, or the like, which, together with the zinc and the liquid of the cell, form a miniature battery and generate currents at the surface of the plate with consequent wasting of the metal, whether the external circuit be open or not. Such chemical action within the cell is termed *local action.* It may in large measure be stopped by amalgamating the surface of the zinc, the fluid amalgam serving apparently to protect the foreign particles in the zinc from contact with the electrolyte.

344. Polarization. — When a current is drawn from a simple voltaic cell having, for example, the elements Zn, Cu, and H_2SO_4, the hydrogen soon forms such a layer over the copper as to make it virtually a hydrogen rather than a copper plate, and in consequence the current ceases. The cell is then said to be *polarized.* Polarization may be reduced or entirely prevented by surrounding the positive plate with a second liquid which will be decomposed by the hydrogen. A large variety of cells have been devised in recent years, the most important of which will be noted in the following articles.

It will be noticed that each has its merits and defects, and that none of them admits of universal application. With the perfection of the dynamo, the primary battery has ceased to have the commercial importance it once possessed.

345. The Daniell Cell. — The first and in many respects the most important of the constant current cells was devised by Daniell in 1836. A section of this typical two-fluid cell is shown in Fig. 245.

A is a vessel filled with a dilute solution of sulphuric acid or zinc sulphate, in which is placed the negative element,

a zinc cylinder, *Z*, and a porous earthenware jar, *J*, containing a solution of copper sulphate, and the positive element *C*, a strip of metallic copper. The porous septum serves to keep the liquids separate, but does not offer any sensible obstruction to the passage of the hydrogen toward the copper plate. Now, as it is freed, instead of collecting on this plate it decomposes the copper sulphate, freeing the metal, which is deposited on the copper plate. The chemical reaction may be represented by

$$H_2 + CuSO_4 = H_2SO_4 + Cu.$$

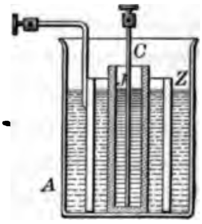


Fig. 245.

An equivalent form of Daniell's cell, known as the gravity battery, has been extensively used for telegraphic purposes. The copper plate, with some crystals of copper sulphate, is placed in the bottom of a jar, and covered to the depth of two or three inches with a solution of the same salt. The jar is then filled with a solution of zinc sulphate, and the zinc, in the form of a branched casting, is suspended in it near the top. If the jar is undisturbed, the liquids will remain separate on account of their difference of density. The advantages of the Daniell cell are the great constancy of its potential difference, or electromotive force, as it is frequently called, at 1.08 volts while supplying a current, and the cheapness of its materials. Its disabilities are a comparatively high internal resistance, a tendency of the copper to deposit on the porous cup, and a diffusion of the copper sulphate toward the zinc plate.

346. The Grove Cell. — In the cell devised by Grove the positive element, a thin sheet of platinum, and the depolarizer, strong nitric acid, are contained in a porous cup, which is surrounded by dilute sulphuric acid and the negative

element, an amalgamated zinc plate. The reaction within the porous cup may be represented by

$$3\,H_2 + 2\,HNO_3 = 4\,H_2O + N_2O_2.$$

The nitric oxide, on coming to the air, takes up another molecule of oxygen, forming the familiar red peroxide.

The advantages of this cell are a high E. M. F. (about 1.9 volts), a very low internal resistance, and entire absence of polarization. It has the serious drawback of emitting the noxious fumes of nitrogen tetroxide, and must be kept under a hood. Besides the fact that it is necessary to take the cell apart and wash it thoroughly every time it is used, the original cost of the cell is considerable.

347. Bunsen Cell. — The platinum in Grove's cell is sometimes replaced by a stick of carbon, in which form it is known as the Bunsen cell. Since, however, the increase of size necessitates the use of a larger amount of nitric acid, the change is one of doubtful utility. Another change, due also to Bunsen, — the substitution of a chromic solution for the nitric acid, — is a distinct gain in the direction of economy. When potassium bichromate, $K_2Cr_2O_7$, is treated with strong sulphuric acid, chromium trioxide, CrO_3, one of the best oxidizing agents known, is formed. By its action the nascent hydrogen in the cell is reduced to water and prevented from collecting on the positive pole.

In the ordinary form of the bichromate cell the porous cup is entirely dispensed with, and both the zinc and the carbon are dipped into the same solution. This cell possesses a very high electromotive force (2.1 volts), and is free from noxious fumes; but, as the chromic acid attacks the zinc, it is necessary to remove the latter when not in use. In the last-mentioned form the cell is far from a constant one.

348. Leclanché Cell. — The elements of an important form of cell devised by Leclanché are zinc and carbon; the electrolyte is sal-ammoniac, and the depolarizer oxide of manganese. In the original construction the MnO_2 was mixed with powdered carbon and packed in a porous cup, which surrounded the positive element. In the more recent forms the manganic oxide and the carbon are pressed together into a stick and introduced into the NH_4Cl together with the zinc. The advantages of this cell are a considerable potential difference, say 1.5 volts, and entire freedom from local action. If tightly sealed, it may be left years without attention. In the single fluid form it polarizes rapidly, but is of the highest value for the purposes of signaling where it remains on open circuit, except for the brief period during which it is pressed into service.

349. Clark Cell. — The voltaic cell now recognized as best suited to furnish a standard of potential difference, was first made and studied by Latimer Clark in 1873.

The elements are zinc and mercury, and the electrolyte mercurous sulphate mixed with a saturated solution of zinc sulphate. The form of this cell, recommended by the National Academy of Sciences, and adopted by act of Congress for the purpose of the legal definition of the volt, is shown in Fig. 246. The containing glass vessel consists of two tubes, *A*, *B*, not less than 2 cm. in diameter and 3 cm. long, joined to a common neck and fitted with a glass stopper. At the bottom of each tube a wire, 0.4 mm. diameter, is sealed into the glass. This wire in one leg, *B*, is covered with pure mercury, and the other, in *A*, with an amalgam consisting of 90 parts mercury to 10 parts zinc. A layer of a paste, about 1 cm. thick, made of mercury, mercurous sulphate, and crystals of zinc sulphate moistened with a solution

of zinc sulphate, is placed above the mercury in the leg *B*. The paste and the amalgam are then covered with a layer of crystals of zinc sulphate, about 1 cm. thick, and the whole vessel filled with a saturated solution of zinc sulphate. The stop-

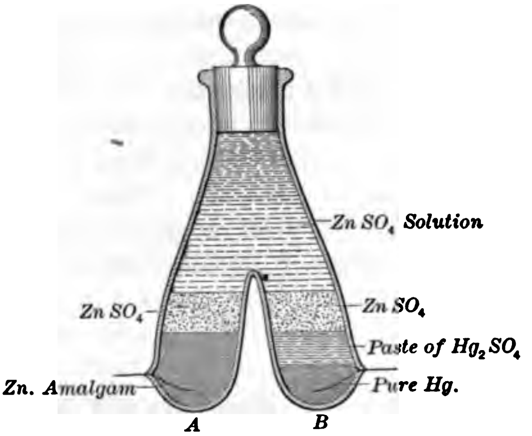


FIG. 246.

per, having been brushed with a solution of shellac, is pressed firmly in place. The potential difference of such a cell between the temperatures 10° C. and 25° C. is accurately given by

$$\text{P.D.} = 1.434 \overset{\text{volts.}}{} \{1 - 0.00115\,(t - 15°)\}.$$

350. Definition of Current Strength. — The strength of a current in a conductor is defined as the quantity of electricity which passes any cross section per unit time. If Q denote the total and constant flow across any section of a conductor, t the time, and i the strength, or intensity, of the current,

$$i = \frac{Q}{t}. \tag{1}$$

351. Practical Unit of Quantity and Current. — For the purpose of measuring electrical currents, the unit of quantity, defined in Art. 272, is found to be inconveniently small. Accordingly, a new unit, called the *coulomb*, has been selected.

It may be defined as

$$3\,(10)^9 \text{ C. G. S. electrostatic units of quantity.}$$

This leads at once to the practical unit of current, or *ampere*, which may be defined as that current in which one coulomb of electricity per second flows across any section of the conductor.

352. Ohm's Law. — When a constant difference of potential is maintained between two points of a circuit, the current which passes every intermediate point is found to be exactly proportional to the potential difference. This result, first announced by Ohm in 1827, is known as Ohm's Law.

If i denote the current and E the difference of potential, this law may be written

$$i = kE, \tag{2}$$

where k is a physical constant which depends only on the nature and dimensions of the conductor, if the temperature remains constant, and is independent of the value of the electromotive force.

Ohm wrote

$$i = \frac{E}{R}, \tag{3}$$

where $R = \frac{1}{k}$ is called the resistance.

It is of interest in this connection to remark that Cavendish, in some unpublished experiments made in 1781, antici-

pated the law of Ohm by showing that, if the resistance was a function of the current, the exponent in the relation,

$$R = ai^n,$$

could not exceed 0.02, a being a mere number.

This result is the more remarkable because it was obtained thirty years before the invention of the galvanometer, and with no other means of judging the current than receiving the shock of a Leyden jar when discharged through a poor conductor.

353. Definition of the Ohm. — The practical unit of resistance, called the *ohm*, may be defined as the resistance of that conductor in which the potential difference of one volt between its terminals would generate a current of one ampere. Thus,

$$1 \text{ ampere} = \frac{1 \text{ volt}}{1 \text{ ohm}}.$$

The legal definition of the ohm is as follows: "The resistance offered to an unvarying electric current by a column of mercury at a temperature of melting ice, 14.4521 grams in mass, of a constant cross-sectional area, and of the length of 106.3 centimeters." The cross section of this thread is very nearly 1 square millimeter.

A further discussion of the practical units and their dimensions will be found in Art. 421.

354. Electrolysis. — When a fluid on being subjected to a definite potential difference is separated into two components, the process is called *electrolysis*. Any compound liquid which is capable of such separation is termed an *electrolyte*. The two components, called *ions*, into which it is divided ap-

pear, however, only at the electrodes. That component which appears at the anode is called the anion, or electro-negative component; likewise, that component which appears at the kathode is called the kation, or electro-positive component.

If, for instance, two electrodes of platinum be placed in hydrochloric acid and connected to the poles of a battery, hydrogen will be liberated at the negative electrode and chlorine at the positive.

The phenomenon of electrolysis was discovered accidentally by Carlisle and Nicholson in the year 1800. On learning of Volta's invention of the pile, these experimenters set to work at once to construct one from half-crown pieces, and, thinking to make a better contact at one point of the circuit, used a drop of water to connect the wire with a metal plate. When the circuit was closed, bubbles were observed to rise through the liquid, and the odor of hydrogen gas was detected. Further investigation showed that, whenever a current was passed through water, hydrogen was given off at the negative and oxygen at the positive electrode.

355. Laws of Electrolysis. — The fundamental laws of electrolysis, established by Faraday, are as follows: The mass of the ions liberated at either pole is proportional

1°, to the quantity of electricity which passes any section of the circuit.

2°, to the chemical equivalent of the ion, by which is meant its atomic mass divided by its valence in the compound electrolyzed.

Thus, let q stand for the quantity of electricity which passes any point of the circuit, m the total mass liberated, A the atomic mass of the ion, and V its valence. Then

$$m = kq\frac{A}{V}, \tag{4}$$

where k is a constant depending only on the units employed. If t denote the time of flow of the steady current i, equation 4 may be written

$$m = \frac{kA}{V} it = \epsilon i t, \tag{5}$$

by writing $\epsilon = \frac{kA}{V}$. ϵ is called the *electro-chemical equivalent.*

It is found by experiment that one ampere, flowing for one second, deposits 0.001118 gm. of silver. Taking the atomic mass of silver as 108, that is, $O = 16$, the value of k is found to be 1.035, from which the electro-chemical equivalents of the other elements may be calculated.

Thus, by equation 5, if the atomic mass of copper be taken as 63.64, and its valence as 2, the electro-chemical equivalent of copper is

$$\epsilon = 0.0003294 \frac{\text{gm.}}{1 \text{ ampere} \cdot 1 \text{ sec.}}.$$

356. Theory of Electrolysis. — The facts of electrolysis are best accounted for by the theory originally proposed by Grotthuss and developed by Clausius. Each molecule of an electrolyte is assumed to be composed of two parts which are oppositely charged, and united by certain bonds usually designated as chemical. These parts may consist either of single atoms as in HCl, or of a group of atoms, which in many reactions behaves like an elementary substance, in that the group may be transferred from one compound to another without loss of identity. NO_3 and SO_4 are examples of such groups or compound radicals. It further seems probable that the positive ion of the molecule does not remain united to the same individual negative ion, but that there is a frequent change of partners, and that during this interchange some of the ions may remain for a longer or shorter time disconnected from the other portion of the chemical

molecule. This assumption is justified by the fact pointed out in Art. 197, namely, that the change of the freezing point and the vapor pressure in solutions of strong bases and acids was such as to indicate the presence of a number of molecules greater than that calculated from other data.

Important evidence that continual interchange in the parts of the molecule takes place is furnished by the behavior of the solutions of strongly combined salts in forming compounds in which the uniting bonds are much weaker. If the weaker compound is sufficiently insoluble, it appears as a precipitate, and justifies the conclusion that a quantity of the corresponding compound is present in every case, though from its solubility it may not be so readily detected.

Suppose, now, that any compound liquid, such as hydrochloric acid, is subjected to a difference of potential, by being placed between the electrified plates *P*, *N* (Fig. 247). The first effect will be to face all the molecules about, so that the negative or chlorine atom will be toward the positive electrode, and the hydrogen toward the negative electrode. If, now, a molecule close to the positive electrode were to be broken up, and this condition can be more readily effected on account of the stress produced by the field, the chlorine would be attracted to *P*, and might give up its charge, while the hydrogen would start to move toward the other electrode. Before it has gone far it may be supposed to meet a free chlorine atom, with which it pairs off. On being again set free it will make another step toward its goal, the negative electrode, which it will ultimately reach. In the same way the free chlorine atoms will gradually work their way toward the positively charged plate.

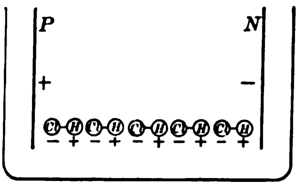


Fig. 247.

There are thus two material currents of ions constantly moving in opposite directions through the liquid, maintaining corresponding convection currents of electricity.

The part of this theory which is most open to objection is the rather arbitrary way in which the ions are made to give up their charges, and the assumption that every monad atom, whatever its nature, carries the same charge, every dyad atom or ion one-half as much, every triad atom one-third of this amount, and so on. Nevertheless, the theory is a most valuable aid in the statement of the facts of electrolysis.

If the potential difference between the electrodes does not exceed a certain minimum amount, there is observed a feeble current through the liquid for a time, but, however, without liberation of the ions, as if the uncombined ions moved up to the electrodes and the process stopped for lack of further dissociation, or that an atom could not be forced into company of its own kind except by an electromotive force exceeding a finite magnitude. The statement that the potential difference must exceed a finite amount does not, however, apply to the case where the anode is the same metal as the kation.

357. Secondary Actions. — In many cases important secondary chemical processes arise from the evolution of the ions, so that the substances liberated at the electrodes are the products of the action of the ions on the solution of the electrolyte. Thus, in the electrolysis of a solution of Na_2SO_4, hydrogen is liberated at the kathode and oxygen at the anode. The initial products of electrolysis in this case are probably Na_2 and SO_4, but as sodium is a highly oxidizable element, it robs the water of its oxygen, forming Na_2O and setting the hydrogen free. At the same time the sulphion at the anode, not being able to exist in the free state, breaks up

into O and SO_3. The oxygen gas is liberated, while the sulphur trioxide takes up another molecule of water, forming H_2SO_4.

When a current is passed through water containing a small quantity of H_2SO_4, the products of electrolysis are exactly the components of water, two volumes of hydrogen and one of oxygen. Water was, accordingly, long regarded as a typical electrolyte. The researches of Kohlrausch have, however, shown that, in proportion as water is freed from impurities, its resistance increases, and that absolutely pure water would doubtless be a non-conductor. The hydrogen is, accordingly, to be regarded as arising from the electrolysis of H_2SO_4 and the oxygen from the decomposition of SO_4.

It is possible, in some cases, to deposit an alloy of two metals. Thus the kathode may be plated with brass from a mixture of the cyanides of zinc and of copper.

The secondary actions in the electrolytic cell are often very complicated and difficult to control. The alkali metals cannot be obtained by using weak solutions of their salts.

358. The Voltameter. — Faraday, after his discovery of the law expressed by equation 5, proposed the use of the electrolytic cell as a current measurer, which he named the *voltameter*. Experience has shown that silver is the best substance to use for deposit in such a cell. The legal definition of the ampere, by act of Congress, is "the practical equivalent of the unvarying current, which, when passed through a solution of nitrate of silver in water, in accordance with standard specifications, deposits silver at the rate of 0.001118 gram per second."

The arrangement prescribed is as follows:

The kathode shall be a platinum bowl, not less than 10 cm. in diameter and 4 cm. deep. Before use it must be washed

successively with nitric acid, distilled water, and absolute alcohol.

The anode shall be a disc of pure silver, supported by a silver rod riveted through the center. To prevent any particles of silver falling on the kathode, the anode must be wrapped with filter paper properly secured. The electrolyte shall consist of a neutral solution of pure silver nitrate, containing about 15 parts, by weight, of the nitrate to 85 parts of water. The external circuit must contain a resistance of at least 10 ohms, and the total current through this voltameter must not be much over one ampere. After the current has been allowed to run about half an hour, the deposit is carefully washed and dried. The current may then be calculated by the formula

$$i = \frac{m}{t\epsilon}, \text{ where}$$

$$\epsilon = 0.001118 \frac{\text{gm.}}{\text{amp.} \times 1 \text{ sec.}}.$$

For the purpose of measuring stronger currents the copper voltameter, consisting of an anode of pure copper in a solution of $CuSO_4$, or $Cu2(NO_3)$, is used. The tendency of copper toward oxidation with strong currents, or toward dissolution in the liquid with feeble ones, renders determinations with the copper voltameter less certain than those made with the silver voltameter.

359. Applications of Electrolysis. — The process of electrolysis has been extensively used in the arts:

1°. In the reduction of metals from solutions of their ores. Copper so deposited is remarkable for its purity.

2°. In electroplating, where it is used chiefly to deposit a layer of gold or silver on the surface of a less precious metal.

The solutions used are generally the double cyanide of potassium and gold or silver, as the case may be. Surfaces of brass are frequently nickel-plated, to prevent tarnishing, and surfaces of steel, to prevent rusting. The latter end will not, however, be attained unless the surface has been previously coated with copper. The layer of zinc on galvanized iron, so called, is not electrolytically deposited, as the name would imply, but is a coating obtained by dipping the iron in melted zinc.

3°. In electrotyping, a method of copying in reversed relief the designs of woodcuts, printing type, engraved plates, etc., in a thin sheet of electrolytically deposited copper. The surface to be copied is first coated with a thin layer of graphite, and then immersed in the copper solution, as the kathode. After being subjected to the action of the current for a sufficient time, a thin sheet of copper is deposited, which reproduces with great fidelity the original design of the mold. This sheet is strengthened by filling the back with melted type metal. Such plates may then be used to print from, and will afford about 80,000 impressions. Most books which run through large editions are printed, not from the original type, which would wear rapidly, but from copper-plate copies of the page.

360. Secondary Cells. — After a current has passed through an electrolytic cell, consisting, *e.g.*, of two platinum electrodes and acidulated water, the plates become coated respectively with a layer of the liberated gases, so that they are no longer electrically equivalent to platinum, but are virtually plates of hydrogen and oxygen. As there are then present all the essentials of a voltaic cell, namely, two dissimilar conductors and an electrolyte, the electrodes, if disconnected from the original source of current, will show a reverse electromotive

force, and will give a very brief current in the opposite direction on being connected by a conductor. Electrolytic cells used in this manner as a source of current are called *secondary* or *storage batteries*.

By using electrodes which may be chemically changed, the energy stored in such a cell may be much increased. Planté, in 1860, devised a cell consisting of two plates of lead immersed in sulphuric acid. By the passage of the primary current, the surface of the anode was coated by a film of lead dioxide, and that of the kathode reduced to a spongy metallic state in which it is chemically very active. On connecting the electrodes for the discharge, the hydrogen liberated at the positive plate reduces the dioxide to monoxide, while the oxygen, going to the negative plate, forms a similar layer of oxide there. The potential difference of such a cell is from 1.8 to 2 volts.

Faure, in 1881, modified the Planté cell by rolling up the lead plates with a layer of red lead and flannel between them. In charging, the red lead at the anode is peroxidized, while that at the kathode is reduced to a lower oxide, and finally to the metallic state. By this arrangement a greater layer of the working substance is obtained, and less time is required to "form" the plates. In the more modern forms, the red lead, or some equivalent paste, is pressed into holes in a lead grating or grid, where it is held by a suitable attachment. Secondary cells have a wide field of application and their actual efficiency is high, but the present cost of construction and maintenance is so great as to impair their commercial efficiency when employed on a large scale.

EXAMPLES.

1. What is the value of the current which would deposit 0.156 gm. of silver in 3 min. 33 sec.? *Ans.* 0.655 ampere.

2. How much water should be decomposed by a current of 0.95 ampere in 58 min.? *Ans.* 0.308 gram.

3. The current through an incandescent lamp is 0.682 ampere, and the potential difference 87.6 volts. What is the resistance of the lamp? *Ans.* 128 ohms.

4. What current will a battery of 8 Daniell cells produce through an external circuit having a resistance of 5.7 ohms, the resistance of each cell being 3.4 ohms, and the E. M. F. 1.08 volts.? *Ans.* 0.263 ampere.

5. A dynamo with internal resistance of 11 ohms produces an E. M. F. of 830 volts. How many 10-ampere arc lamps in series will such a machine supply if the resistance of each lamp is 4.5 ohms? *Ans.* 16 lamps.

CHAPTER XXII.

THE ELECTROMAGNETIC FIELD.

361. Oersted's Discovery.—It had been known since 1676 that there existed some connection between electricity and magnetism, for compass needles had been reversed in polarity during thunderstorms, and steel wires magnetized by the discharge of Leyden jars, but the effects observed were too erratic to permit any systematic conclusions to be drawn from them. The first definite relation was discovered by Oersted in 1819, who found that when a conductor conveying a current was presented to a magnetic needle, the latter turned so as to set itself at right angles to the conductor; and, further, that if the magnet be moved around the wire the same end would always point forward.

362. Magnetic Field Due to a Current in a Straight Conductor. — In terms of the magnetic field, Oersted's discovery is equivalent to the statement that the lines of magnetic force produced by a current in a straight conductor, so arranged that the other portions of the circuit are at a great distance, are circles in a plane at right angles to the conductor, their direction being related to that of the current as the direction of rotation is to the advance of a right-handed screw. Thus, if A (Fig. 248) represent the end of a conductor into which a current is flowing, then the lines of force will have the direction indicated in the figure.

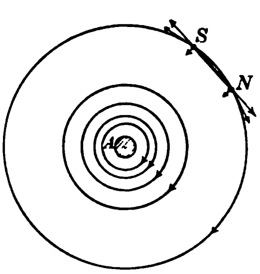


FIG. 248.

The existence of such a field may be shown experimentally by passing a stout copper wire through a piece of cardboard and scattering iron filings over the latter. If the card be gently tapped while a strong current is sent through the wire, the filings may be seen to arrange themselves about the wire in concentric circles which contract as the tapping continues. This movement of the filings across the field is an example of the motion of a paramagnetic body from places of weaker to those of stronger force.

If the force on each of the poles N, S of the magnet in Fig. 248 be resolved into components parallel and perpendicular to its axis, it will appear that the magnet as a whole will not be urged in either direction parallel to the line of force, but that there will be a resultant attraction perpendicular to this direction, which will draw the magnet toward the conductor.

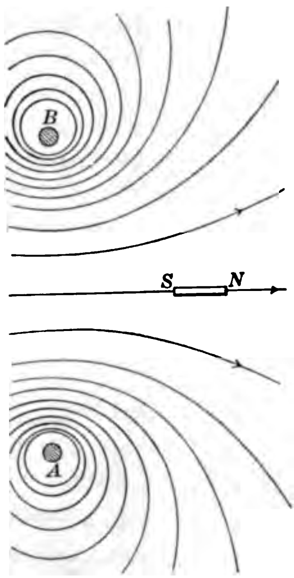


Fig. 249.

363. Field Produced by a Current in a Circular Coil.—The field produced by a current flowing through a wire bent into the form of a circle is shown in Fig. 249. If the current is supposed to enter the plane of the paper at A and leave it at B, then a magnet, NS, placed in the axis of the coil would set itself so that the north pole would point to the right in the figure.

It may be shown, from theoretical considerations alone, that the field produced by a current flowing through a cir-

cuit is identical with that which would be produced by a magnetic shell uniformly magnetized perpendicular to its surface and bounded by the circuit, except that this statement must not be applied to a point within the substance of the shell. The magnetic moment of the circuit may be found by multiplying the area of the circuit by the strength of the current, and the direction of the magnetization by the familiar right-handed screw rule. Thus, in the figure, if a screw were rotated in the direction the current flows, it would advance toward the north face of the shell.

364. Ampère's Theory of Magnetism. — The equivalence in magnetic action of a small plane circuit, at distances which are great compared to the dimensions of the circuit, and of a magnet whose axis is perpendicular to the plane of the circuit, was first shown by Ampère, who perceived that all magnetic phenomena could be explained in terms of electric currents. He, accordingly, propounded the hypothesis, now generally accepted, that each molecule of a magnetic substance has a current circulating in it. Since magnetism does not appear to vary with the time, it is necessary to suppose that the molecular currents flow without resistance. As Ampère's theory is not inconsistent with any known facts, all the laws of magnetism might be derived from the conception of a small closed circuit, though such a method would evidently lack much of the simplicity and directness of that employed in Chapter XX.

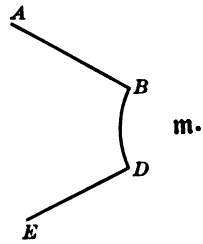


Fig. 250.

365. Force Exerted by a Current on a Pole. — Let *ABDE* (Fig. 250) be a circuit having the portion *BD* bent into the arc of a circle, while the portions *BA* and *DE* extend in radial directions to a great distance.

Call r the radius of the circle,

l " length of the arc BD,

$\mathfrak{m}$ " strength of the pole placed at the center; then if a current, i, be sent through the circuit in the direction BD, experiment has shown that the north pole $\mathfrak{m}$ will be acted on by a force directed up from the paper, which may be written

$$F = k\frac{\mathfrak{m}il}{r^2}, \tag{1}$$

where k is a constant depending on the units employed.

366. Rowland's Experiment. — The effect on a magnetic needle of a convection current, produced by moving a charged conductor, was investigated by Rowland in 1876. The apparatus consisted, essentially, of a gilded ebonite disc maintained at a high potential and revolved beneath a sensitive magnetic needle which was carefully shielded from air currents and electrostatic influences. When the needle was suspended parallel to the circumference of the rapidly revolving disc, it was deflected to the right or left according to the sign of the charge, showing that a convection current produced the same magnetic effects as a current in a wire.

367. Electromagnetic Unit of Current. — Since equation 1 is an independent equation, it may be used to obtain a new definition of current strength. Thus, taking $k = 1$, and giving the other quantities their usual unit values, i also becomes unity. This new, or electromagnetic unit of current, expressed in words, is that current which exerts the force of one dyne on the unit magnetic pole placed at the center of a circle having a radius of one centimeter, when flowing through an arc of this circle one centimeter long. The ampere or practical unit of current is one-tenth of this electromagnetic unit.

368. Circular Coil in a Uniform Field. — By the law stated in Art. 363, that any closed circuit is equivalent to a magnetic shell having a magnetic moment equal to the strength of the current multiplied by the area of the circuit, the turning effect experienced by a plane circuit in a uniform field may be written down at once. Let H be the strength of the field, A the area of the coil, i the strength of the current, and θ the angle made by the north face of the coil with the lines of force (Fig. 251). Then the mechanical moment experienced by the coil will be

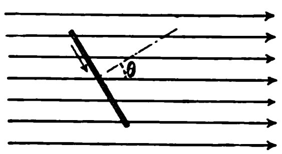


Fig. 251.

$$M = HAi \sin \theta. \tag{2}$$

The fact that any small circular current will, after the analogy of a magnetic shell, turn in a magnetic field so that its positive or north side will face along the lines of force may be differently expressed, thus: Every electric circuit will turn so as to include a maximum number of lines of force. In applying this rule the lines of the field which pass through the coil in the same direction as those produced by the current are to be reckoned positive, and those which pass in the opposite direction negative.

369. Force Experienced by a Conductor in a Magnetic Field. — The advantage of the above method of treating the motion of a small coil in a magnetic field is that the rule just derived is applicable to all circuits, whether movable in parts or as a whole. The generalization is due to Maxwell, who stated it essentially as follows: Every conductor conveying a current, when placed in a magnetic field, experiences a force urging it in such a direction as to increase the total induction through the area bounded by the circuit.

This principle is illustrated in the apparatus shown in Fig. 252. *GP* is a movable conductor supported on a pivot, *G*, resting in a mercury cup, *E*, and dipping into a trough of mercury at *P*. A wire, *ZA*, joins the battery *B* in circuit with the cup and the trough. The current starting from the battery enters the mercury trough at *D*, where it divides, part going through the circuit *DRP* and part through *DQP*. These currents again unite at *P*, and return to the battery by the path *PGEZ*. The lines of force due to the current from *P* to *G* run down through the area bounded by the circuit *DRPG*, and upward through the area bounded by *DQPG*. If the induction due to the external magnetic field be downward through the area *PRDQ*, then the movable arm *PG* will experience a moment turning it from *P* toward *Q*, since by such motion the induction through *DRPG* is increased by the addition of positive lines, and that through *DQPG* is increased by the removal of negative lines.

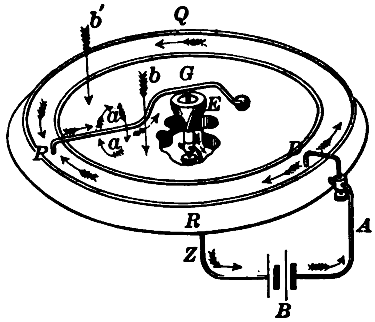


Fig. 252.

Fig. 253.

370. Electromagnetic Rotations. — The problem of continuous rotation of a portion of a circuit about a magnet, or, in other words, the continuous conversion of the energy of an electric current into available mechanical work, was first solved by Faraday in the simple device shown in Fig. 253, which consists of a tube stopped at the bottom by a cork covered with a layer of mercury and pierced by a magnet,

NS. The stopper at the other end carries a hook, from which is suspended a wire, *PQ*, dipping into the mercury. On passing a current through it by means of the connections shown, *PQ* will revolve about the pole of the magnet in a manner explained in the preceding article. A great variety of other forms of apparatus have been devised, showing the rotation of a portion of a circuit about a magnet, or about another portion of the circuit, and the rotation of a magnet about its axis when included in the circuit. They are all variations on the theme of Art. 369 and do not call for further explanation. Their chief interest is the place they occupy in the history of the development of the electric motor.

371. Solenoids. — If a number of small, like circuits, all facing one way, be placed side by side on a common axis, they will form an apparatus whose external field does not differ sensibly from that which would be produced by a magnet built up of a series of magnetized discs. Such a device is known as a *solenoid*. This construction is most conveniently approximated by coiling a wire into the form of a helix (Fig. 254).

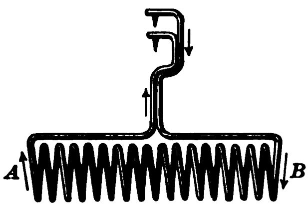


Fig. 254.

If the current be made to traverse the coils in the direction indicated, the end *A* will behave like the north pole of a magnet, and *B* like a south pole. There is, however, this distinction between a magnet and a solenoid, that within the magnet the lines of force, defined as in Art. 316, run from north to south, while within the helix they have the same direction as the induction, running from south to north.

372. Electromagnet. — If the space within a helix be filled with a permeable medium, such as soft iron, the lat-

ter will become magnetized by induction when the current passes, forming what is known as an *electromagnet*. On breaking the current, the magnetism nearly disappears if the iron be soft. The relation between the direction of the current and the polarity of the magnet is obviously the same as that of the solenoid.

The most useful form of the electromagnet is one in which the core has a U, or horseshoe, form (Fig. 255). On closing the circuit at some point, which may be at any distance from the apparatus, the magnet will attract the armature KK'. If the core of the electromagnet is very soft, and the armature is not allowed to come into close contact with the poles, the iron will be demagnetized as soon as the current is broken, and the armature will fall back. If, however, the bar be permitted to come into intimate contact with the poles of the magnet, the residual magnetism will be so great that considerable force may be found necessary to detach the armature. The explanation is that, when the iron forms a continuous magnetic circuit, the molecules, in the strong field produced by the current, are oriented so that all face the same way, as one follows along any line of force, and each little magnetic molecule is constrained by the action of its neighbors, even after the original inducing force is removed. If, however, this arrangement is once broken by forcing off the armature, the magnetism of the core falls to an indefinitely small value depending on the retentiveness of the iron. The effects just noted also indicate why it is desirable that every permanent magnet should have its poles connected, when not in use, by a bar of soft

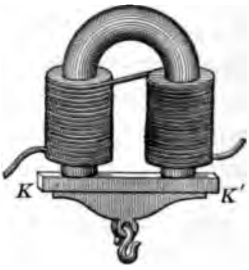


Fig. 255.

iron, or keeper, for in this way the original setting of the molecules in the hardened steel is better guarded from derangement by the action of demagnetizing forces.

When a magnet is inside a helix bearing a current, it is subject to a force in the direction of the axis, unless the arrangement of the helix and magnet is a symmetrical one. The conclusion holds true, also, if the magnet owes its magnetism to the action of the current alone; hence, a movable soft iron core will be drawn into a helix when made a part of a closed circuit. The mechanical advantage of such an arrangement is the readiness with which greater range of motion is secured, so that it is frequently employed in automatic regulating apparatus. Illustrations of its use may be seen in Figs. 260 and 278.

373. Tractive Force of a Magnet. — The weight which can be supported by an electromagnet is limited only by the dimensions of the apparatus. If

$F =$ the force in dynes,
$B =$ the induction,
$A =$ area of contact at the pole face,

it may be shown that

$$F = \frac{B^2 A}{8\pi}.$$

EXAMPLES.

1. If a current of 5.7 amperes flows through a coil 48 cm. in diameter, consisting of 3 turns of wire, what will be the strength of the field at the center? *Ans.* 0.448 $\text{gm.}^{\frac{1}{2}} / \text{cm.}^{\frac{1}{2}}$ sec.

2. What must be the radius of a single coil which, when traversed by a given current, would produce a field at the center equal to that produced by the same current sent through two circular concentric coils of 15 cm. and 45 cm. radius, respectively, and joined in series? *Ans.* 11.2 cm.

CHAPTER XXIII.

GALVANOMETRY.

374. Galvanometer. — The term *galvanometer* is commonly applied to any instrument used to measure or detect the existence of a current by a deflection produced in a magnetic needle or its equivalent. If i denote the current and θ the deflection of the needle, the value of i may be written

$$i = Kf(\theta), \tag{1}$$

in which K is a constant, and $f(\theta)$ has a simple form, usually a circular function of the angle, or the angle itself. When K may be determined from the dimensions of the instrument and the strength of the earth's field alone, the apparatus is called an absolute galvanometer.

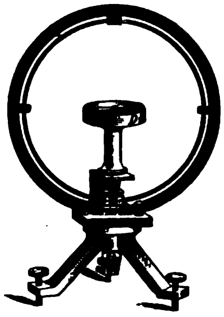


Fig. 256.

375. Tangent Galvanometer. — The tangent galvanometer, so called because the current is calculated from the tangent of the angle of deflection, consists of a short needle suspended at the center, or on the axis, of one or more coils of wire exactly circular in form (Fig. 256). The ends of the coils to which connection is made are brought close together, and bent at right angles to the plane of the coil so that they shall exert no influence on the needle.

To find an expression for current strength in this instrument, suppose that AB, the plane of the coil (Fig. 257), is

set parallel to the magnetic meridian, A being toward the north. Also suppose that the current is traversing the coil in such direction that the north pole, m, of the needle is urged to the right, and the south pole, m′, to the left.

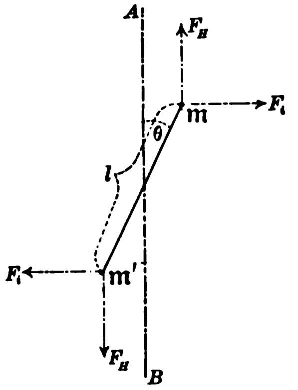


FIG. 257.

Let R = radius of the coil,

i = current in electromagnetic units,

m = strength of each pole of the needle,

l = length of the needle,

θ = deflection from plane of the meridian,

F_H = force experienced by each pole in the earth's field,

F_i = force experienced by each pole in the field of the coil,

M_H = moment experienced by the needle due to the earth,

M_i = moment experienced by the needle due to the coil,

H = horizontal component of the earth's field.

Then, by equation 2, Art. 317,

$$(2) \qquad F_H = H\mathrm{m},$$

whence

$$(3) \qquad M_H = F_H l \sin\theta = -H\mathrm{m}l \sin\theta.$$

Assuming that m is at the center of the coil,

$$(4) \qquad F_i = i\frac{\mathrm{m}2\pi R}{R^2}, \text{ by equation 1, Art. 365,}$$

whence

$$(5) \qquad M_i = F_i l \cos\theta = \mathrm{m}l\frac{2\pi i}{R}\cos\theta.$$

If the needle is in equilibrium,

$$M_i + M_{ii} = 0; \tag{6}$$

equating and solving for i,

$$i = \frac{HR}{2\pi}\,\frac{\sin\theta}{\cos\theta} = \frac{HR}{2\pi}\tan\theta. \tag{7}$$

It thus appears that current strength does not depend on the magnetic constants of the needle if the dimensions of the latter are so small that both poles may be considered as at the center of the coil. When used to measure large currents, the coil of the galvanometer consists of a single turn of heavy copper wire. If feeble currents are to be measured, the coil is made of many turns of fine wire. In this case, since F_i is proportional to the length of the wire, or to $2\pi nR$, where n represents the number of turns, the expression for the current becomes

$$i = \frac{RH}{2\pi n}\tan\theta. \tag{8}$$

This equation is also written

$$i = \frac{H}{G}\tan\theta = K\tan\theta;$$

$G = \frac{2\pi n}{R}$ being called the true constant of the galvanometer and K the working constant. When great accuracy is desired, corrections must be made for the width and depth of the coil and the length of the needle.

It should be noted that the value of i in each equation of this article is in electromagnetic C.G.S. units, and that to reduce it to amperes it must be divided by ten.

376. The Sine Galvanometer. — If, instead of setting the coil of Fig. 256 parallel to the magnetic meridian, it had been

turned until its plane coincided with the needle while the current was running, M_i would have lacked the term cos θ. The value of i would then have been

$$i = \frac{H}{G} \sin \theta ;$$

θ being the angle between the needle and the meridian.

This form of instrument is not as convenient in use as is the tangent galvanometer, and it is little used.

377. Sensitive Galvanometers.—*Astatic System.* Great sensitiveness in a galvanometer intended to detect small currents may be secured by combining two or more strongly magnetized needles in an *astatic* system, *i.e.* one so arranged that it would experience a very small directive force in a uniform field, and suspending it so that different parts are affected by different coils of the same wire placed very close to the needles.

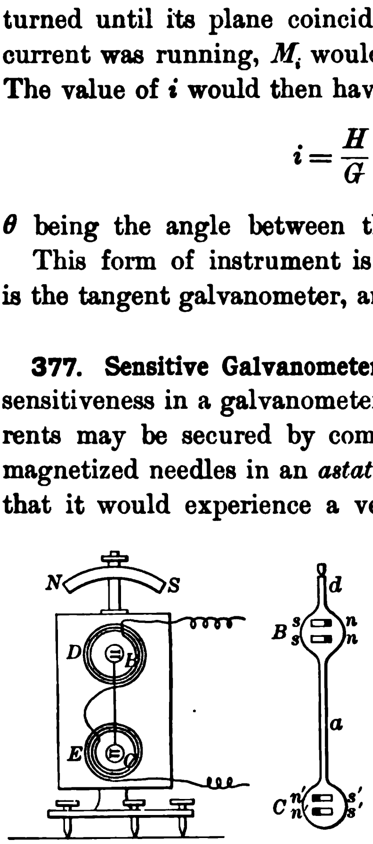


FIG. 258.

One type of such galvanometer is illustrated in Fig. 258. The needle system consists of an aluminum strip, *ad*, carrying two discs, *B*, *C*, to the backs of which are fastened pieces of magnetized watch spring, *ns* and *n′s′*, the upper pair having their poles opposite to those of the lower pair. Such a system, provided the magnets are all alike and in the same plane, would be astatic, that is, would remain in equilibrium in all azimuths in a uniform field. A position of stable equilibrium is secured by a control magnet, *NS*, placed above the coils. The needle is suspended by a silk

fiber, so that the discs hang in the centers of two coils, D and E, wound in opposite directions. The face of one of these discs, B, carries a silvered mirror, and the deflection of the needle may be read with a telescope and scale. The vibrations are damped by the motion of the disc in a small air cavity. Such instruments are most often used to detect feeble currents, but may be made to give numerical values by the formula

$$i = K\theta,$$

provided θ is small and K can be found. The usefulness of astatic instruments is seriously impaired by their sensitiveness to changes in the external field incident to the employment of strong electric currents in the vicinity of most buildings.

D'Arsonval Type. Another method of constructing a sensitive galvanometer is to suspend the coil in an intense magnetic field.

An example of this type, commonly known as the d'Arsonval galvanometer, is sketched in Fig. 259. The coil C is suspended between the poles of a strong permanent magnet, NS, by a strip of phosphor-bronze, which serves also to connect the coil with the external circuit. The deflections produced by the current are read by observing through a telescope the image of a scale of equal parts in the mirror M. On breaking the circuit, the coil is restored to its initial position by the torsional elasticity of the suspension.

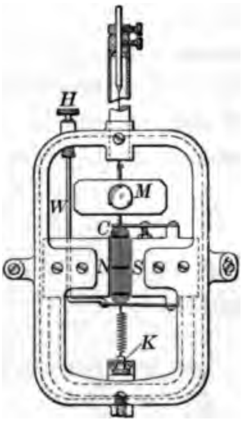


FIG. 259.

This type of instrument possesses three valuable features.

1°. It is practically independent of small variations in the external field.

2°. Its oscillations may be rapidly damped by the currents induced in the coil (see Art. 414).

3°. It may be made very sensitive by increasing the windings and the strength of the field magnets.

By the use of a shunt or divided circuit (Art. 390) the range of currents for which a sensitive galvanometer can be used may be very much extended.

378. Ballistic Galvanometer.—If the duration of a transitory current is very brief compared with the period of the needle of the galvanometer, and there is little damping, the apparatus may be used to measure the quantity of electricity which has passed through the coils. An instrument designed for this purpose is called a ballistic galvanometer. Let Q denote the quantity of electricity which is discharged through the coil, a the angle of the first swing, and K the constant of the instrument. It may be shown that Q can be calculated from the formula

$$Q = K \sin \frac{a}{2}. \tag{9}$$

If the angle is small and the deflection is read by a telescope and scale, the quantity of electricity may be taken as proportional to the first swing expressed in scale divisions.

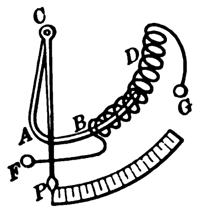


Fig. 280.

379. Direct Reading Galvanometers.—For the measurement of currents in their industrial applications, a variety of galvanometers have been devised which may be placed in the hands of unskilled workmen. The intensity of the current is usually indicated by the position of a pointer over a scale graduated to read amperes. An example of such an instrument, often called an *ampere-*

meter, or *ammeter*, is shown in Fig. 260. *CAB* is a curved piece of iron suspended from the point *C*, and so arranged that it is drawn into the coil *D* when the current passes from *F* to *G*. A pointer, *CP*, moving over an empirically graduated scale, shows, within certain limits, the amount of any steady current which may be passed through the coil. On breaking the circuit, the weight of the moving system returns the pointer to the zero position.

380. Weston Ampere-Meter. — Another type of industrial galvanometer, known as the Weston ampere-meter, con-

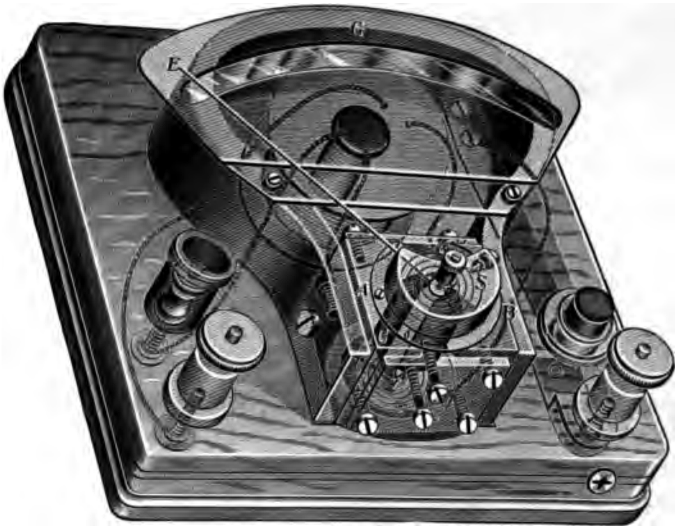


FIG. 261.

structed on the model of the d'Arsonval galvanometer, is shown in Fig. 261. The essential parts of the instrument are a permanent magnet, *AGB*, of great constancy, a fixed soft iron cylinder, *C*, which concentrates the field, and a coil

of wire, *d*, wound about an aluminum frame which turns freely on pivots V, V'. The damping effect of this frame is sufficient to render the motion of the coil dead-beat, *i.e.* aperiodic. *E* is a pointer fastened to the coil and moving over an empirically divided scale, *F*, whenever a current is passed through the coil *d*. On breaking the circuit, the coil is restored to the zero position by spiral springs made of

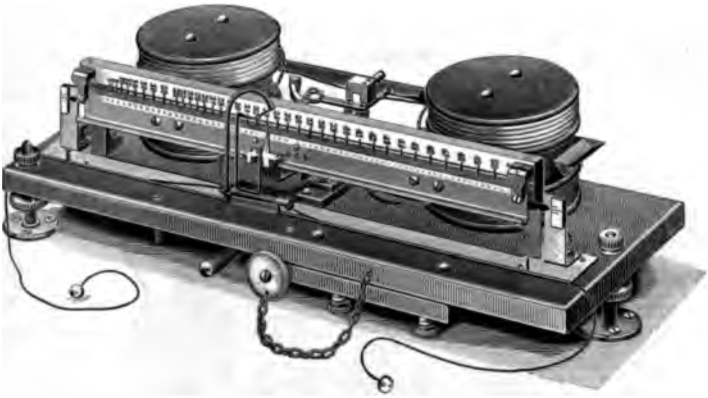


Fig. 262.

a non-magnetic alloy, so as to be unaffected by the field, and placed above and below. Temperature changes in the strength of the magnet are sufficiently compensated by variations in the resistance of the wires.

381. Kelvin's Current Balance. — Lord Kelvin has introduced a type of instrument for the determination of currents, in which the force exerted by one portion of a circuit on another is balanced by a sliding weight. A picture of such an electro-dynamometer is shown in Fig. 262. Its principle may be most readily understood from the diagrammatic sketch (Fig. 263).

A, B, C, D are four horizontal fixed coils, between which is suspended by means of a flexible ligature of fine wires ab, $a'b'$, a light beam carrying the coils M, N. When the current passes, the balance of the arm is disturbed by the forces exerted by the fixed coils on the movable ones, the end N being raised and M depressed. Equilibrium is again established by sliding a counterpoise along the balance arm, which is graduated to a scale of equal parts.

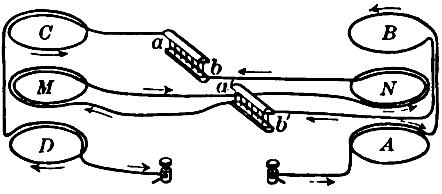


FIG. 263.

It may be shown that the moment of the electromagnetic forces is proportional to the strength of the current in both the movable and the fixed coils, *i.e.* to i^2, and since the moment of restitution of the counterpoise varies as its lever arm, if s denote the number of scale divisions which the counterpoise has been moved from its zero position,

$$(10) \qquad i^2 = k's, \qquad \text{or,} \qquad i = k\sqrt{s}$$

where the value of k, a constant, must be determined by experiment.

382. Potential Galvanometers.—If two points, A, B (Fig. 264), of a circuit through which a current is flowing be connected through a galvanometer having a resistance so high that the current which passes through it does not sensibly affect the disposition of potential in the main circuit, the reading of the galvanometer may be taken as a measure of the potential difference between A and B. If the resistance

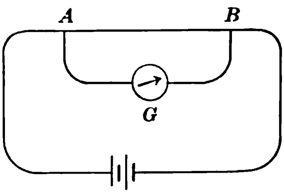


FIG. 264.

R of the galvanometer circuit is known, the fall of potential by Ohm's Law will be

$$(11) \qquad V_a - V_b = iR.$$

Galvanometers calibrated so as to read potential differences directly are known as *potentiometers*.

High resistance current meters of the type shown in Figs. 260 and 261 are extensively used to measure potential differences for industrial purposes. Instruments adapted to this purpose are commonly known as *voltmeters*.

383. Watt-Meter. — Electro-dynamometers are sometimes arranged with low-resistance fixed coils, which carry virtually the whole current, and a high resistance movable coil joined as a shunt to the main circuit, so as to permit the passage of a small quantity of electricity depending on the difference of potential between the points of junction. An arrangement of this sort is shown in Fig. 265. B, B' are the fixed coils carrying the main current, and C, C' the high resistance coils pivoted so as to revolve about a horizontal axis, and joined in parallel with B and B' at m and n. Now, since the deflection is an increasing function of the currents through both coils, it will be a function of the product of the main current and the potential difference between m and n, or, what amounts to the same thing, between the terminals P, Q. Thus, by calibrating the electro-dynamometer empirically, it may be made to indicate the rate at which energy is being supplied. Such an instrument is called a *watt-meter*.

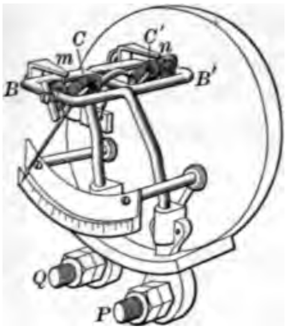


FIG. 265.

384. Resistance of a Conductor. — The numerical definition of electrical conductivity is quite analogous to that of thermal conductivity (see Art. 204).

Thus, suppose that two cross sections in a conductor, at a distance, l, and having an area, A, are maintained constantly at the potentials V_1 and V_2, then the constant electrical force which urges the electricity from places of higher to places of lower potential will be proportional to the rate of change of the potential, *i.e.* to

$$\frac{V_1 - V_2}{l}.$$

Also, since electricity behaves as an incompressible fluid, the total flow Q will vary as the area and as the time; whence

$$Q \propto \frac{V_1 - V_2}{l} At, \tag{12}$$

or,

$$i = \frac{Q}{t} = k\frac{A}{l}(V_1 - V_2), \tag{13}$$

where k is a constant depending only on the nature of the material. Comparing with equation 11,

$$R = \frac{1}{k}\frac{l}{A} = r\frac{l}{A}, \tag{14}$$

where r is called the specific resistance and k the specific conductivity.

From equation 14 it appears that r is the resistance of a unit cube of the substance.

385. Variation of Resistance with the Temperature. — The resistance of the metallic elements, in general, increases with rise of temperature by an amount nearly proportional to the change of temperature.

At 20° C. for most metals, as appears in the table following, the increase per degree is about $\frac{4}{10}$ of one per cent. For alloys the temperature coefficient is considerably less. They are, accordingly, better suited to make standard resistance coils, since for small changes the temperature variations may be neglected.

TABLE OF RESISTANCES.

	Specific Resistance. Ohms to a centimeter cube	Percentage of Variation for a degree at 20° C.
Silver annealed	$1.488(10)^{-6}$	0.377
" hard drawn	1.616 "	
Copper annealed	1.580 "	0.388
" hard drawn	1.616 "	
Gold annealed	2.036 "	0.365
" hard drawn	2.072 "	
Zinc pressed	5.566 "	0.365
Platinum annealed	8.957 "	
Iron "	9.611 "	
Nickel "	12.320 "	
Tin pressed	13.070 "	0.365
Lead "	19.420 "	0.387
Antimony "	35.110 "	0.389
Bismuth "	12.970 "	0.354
Mercury liquid	94.070 "	0.072

A method of estimation of the temperature, from the measured resistance of a platinum wire, has recently been introduced, and affords a means of determining temperatures which are below the range of other thermometers. The apparatus arranged for this purpose is known as a platinum thermometer.

In carbon the change of resistance is in the opposite direction to that in metals; that is to say, it is a decreasing function of the temperature. The specific resistance of electric light carbons is $3.93(10)^6$ C. G. S. units, and the decrease of resistance between 0° and 100° C. is between $\frac{1}{24}$ and $\frac{1}{19}$. The resistance of the filament of a glow lamp when hot is only about one-half what it is when cold.

386. Photo-Electric Properties of Selenium. — In 1873 J. E. Mayhew discovered that the resistance of selenium, which had been carefully annealed, was less when exposed to sunlight than in the dark. Tellurium and carbon are also slightly sensitive to light.

387. Resistance of Insulators. — The resistance of a number of the best insulators is shown in the following table.

SUBSTANCE.	SPEC. RESIST. C. G. S. Units.	TEMP. CENT.
Mica	$8.4\ 10^{22}$	20
Gutta-percha	$4.5\ 10^{23}$	24
Shellac	$9.0\ 10^{24}$	28
Ebonite	$2.8\ 10^{25}$	46
Paraffine	$3.4\ 10^{26}$	46
Glass	greater than any of the above.	

The resistance of insulators decreases very greatly with rising temperature. Some of them, *e.g.* glass, behave as electrolytes as soon as they begin to soften.

388. Resistance Boxes. — For making electrical measurements a number of standard resistances are necessary. These are usually constructed of a copper, zinc, and nickel alloy, and are arranged in sets so that all combinations between the lowest and highest resistance of the group may be

obtained. For this purpose the ends of each coil, after being wound double or non-inductively to prevent sparking and magnetic effects, are connected to a series of brass blocks, A, B, C, D (Fig. 266), and the whole attached to a convenient frame or box. Any of the spaces between the blocks may be closed by the insertion of tapered brass plugs, which have the effect of cutting out of the circuit the resistance of the coils so joined.

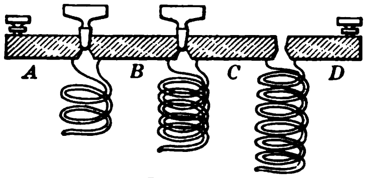


FIG. 266.

Combinations giving all resistances from 1 to 110 are made with the greatest facility if the individual coils have the resistances

$$1, 2, 2, 5, 10, 20, 20, 50;$$

but any required resistance may be obtained with the fewest number of coils when their resistances are in the series

$$2^0, 2^1, 2^2, 2^3, 2^4, \cdots, 2^n.$$

Eight coils in this series, for instance, give all resistances between 1 and 255.

389. Verification of Ohm's Law. — The law $i = \frac{E}{R}$ may be verified within the limit of the errors of observation by either of the following methods:

1st method. Fall of potential along a wire. Let B (Fig. 267) be some constant source of electromotive force, such as a Daniell's cell, which produces a steady current, i, in a wire, AD, of uniform cross section. Let a, b, c, d, e, etc., be a series of points arranged

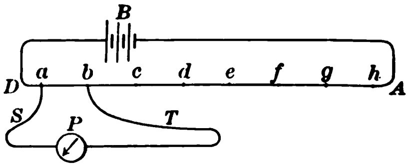


FIG. 267.

at equal distances along the wire, to which may be attached a secondary circuit, ST, containing the potentiometer P. Writing Ohm's Law in the form

$$V_a - V_b = i\frac{r}{A} \cdot L_{ab},$$

it appears that, as i is constant and the wire has a uniform cross section, the fall of potential depends only on the distance between the points a, b, which are joined. If, now, points a, c, a, d, etc., are connected through SPT, the potential differences indicated by the potentiometer will be found to be $2(V_a - V_b)$, $3(V_a - V_b)$ within the limit of errors of observation.

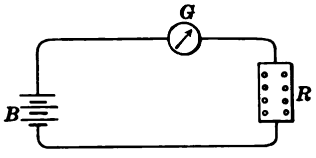


FIG. 268.

2d method. Let B (Fig. 268) be a constant battery, R a resistance box, and G a low resistance galvanometer. If G denote the resistance of the galvanometer, and i_1 the current corresponding to the resistance R_1 in the box, Ohm's Law will be verified if

$$(15) \quad i_1(R_1 + G) = i_2(R_2 + G) = i_3(R_3 + G) = \text{etc.}$$

Writing this in the form

$$i_1\left(\frac{R_1}{G} + 1\right) = i_2\left(\frac{R_2}{G} + 1\right) = i_3\left(\frac{R_3}{G} + 1\right),$$

it is seen that if the ratio of R to G is considerable, say 1000 or more, the fulfillment of the relation $iR =$ constant will be a verification of Ohm's Law within the probable errors of observation.

390. Divided Circuit. — When two points of a circuit having, respectively, the potentials V_a and V_d, are connected

through two conductors, as AC and CD (Fig. 269), having the resistances r_1, r_2, the conductors are said to be joined "in series." The resistance of that portion of the circuit between A and D is

(16) $$R = r_1 + r_2.$$

If the points AD are connected, as in Fig. 270, the conductors are said to be joined "in parallel," or "in multiple arc." The resistance in this case is

(17) $$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2}.$$

To prove it, let I be the total current through the circuit, and i_1, i_2 the respective currents through each branch, then

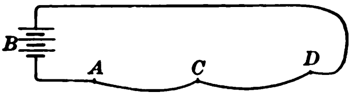


FIG. 269.

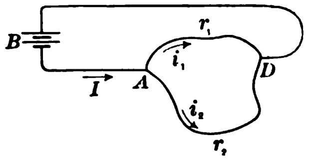


FIG. 270.

since no electricity is supposed to be gained or lost at the points A and D,

(18) $$I = i_1 + i_2.$$

Calling V_a and V_d the potentials of the points A and D, and applying Ohm's Law, equation 18 becomes

$$\frac{V_a - V_d}{R} = \frac{V_a - V_d}{r_1} + \frac{V_a - V_d}{r_2},$$

which is equation 17.

This reasoning, extended to a circuit of any number of branches, gives, evidently,

(19) $$\frac{1}{R} = \Sigma \frac{1}{r}.$$

From the relation

$$i_1 = \frac{E}{r_1},$$

$$i_2 = \frac{E}{r_2},$$

or,

$$(20) \qquad \frac{i_1}{i_2} = \frac{r_2}{r_1},$$

it appears that the current divides at A into parts which are inversely as the resistances.

391. Grouping of Cells. — Suppose that n cells are connected in series (Fig. 271); that is, the positive electrode of the first joined to the negative electrode of the second, and so on. Call the E. M. F. of each cell e, and its resistance r, and the resistance of the external circuit R. Then, as each cell adds the difference of potential e, the whole electromotive force will be ne and the total resistance $nr + R$; hence,

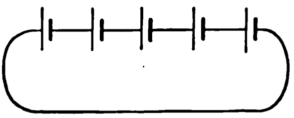


Fig. 271.

$$(21) \qquad i = \frac{ne}{nr + R}.$$

If nr is small in comparison with R, then i will increase in nearly the same proportion as n. But if R is small compared with nr, the current obtained by arrangement in series will never exceed $\frac{e}{r}$, or that from one cell on short circuit.

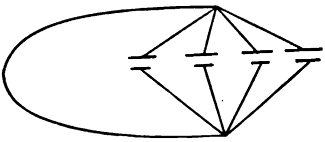


Fig. 272.

If n similar cells (Fig. 272) be connected in parallel, the resistance will be decreased n-fold, but the potential difference will be that of one cell,

namely, e. Hence, the current due to this arrangement will be

$$(22) \qquad i = \frac{e}{\frac{r}{n} + R} = \frac{ne}{r + nR},$$

from which it appears that most will be gained when the external resistance is small.

The maximum current from n cells may be obtained by a combination of the two preceding methods of grouping. Thus, let the cells (as in Fig. 273) be arranged in q rows, in parallel with p cells in each row.

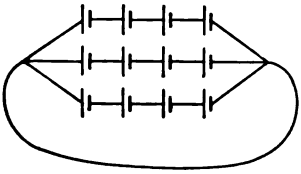


Fig. 273.

By equation 21 the current furnished by one row would be $\frac{pe}{pr + R}$, and by equation 22 the effect of coupling q rows will be

$$(23) \qquad i = \frac{qpe}{pr + qR} = \frac{e}{\frac{r}{q} + \frac{R}{p}}.$$

This may be shown to be a maximum for $\frac{r}{q} = \frac{R}{p}$. Thus, suppose that $\frac{r}{q}$ and $\frac{R}{p}$ are represented by the sides of a rectangle. The area of this rectangle will be constant for

$$\frac{rR}{pq} = \frac{rR}{n} = \text{constant}.$$

But by geometry the square has the least perimeter of all rectangles having the same area; that is, $\frac{r}{q} + \frac{R}{p}$ is least for $\frac{r}{q} = \frac{R}{p}$. Hence, i (equation 23) is a maximum for this value; it is important, however, to note that this does not yield a

maximum efficiency, which always involves a minimum internal resistance of the battery.

392. Wheatstone's Bridge. — The most common method of determining an unknown resistance is by means of an apparatus called Wheatstone's Bridge. The arrangement of the parts is shown diagrammatically in Fig. 274. Four conductors, forming the arms of the bridge *ABCD*, are joined in the order shown. The points *A*, *C*, are connected to the terminals of a battery, *E*, and the points *B*, *D*, by the bridge wire which contains a galvanometer.

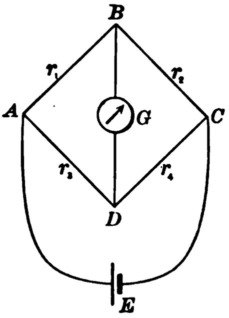


FIG. 274.

In order to find the condition for which the current through *BD* vanishes, call the resistances of the arms r_1, r_2, r_3, r_4, and the corresponding currents i_1, i_2, i_3, i_4.

Let V_a V_c be the potentials of the points *A* and *C*. If there is to be no current through *BD*, the potential at *B* and at *D* must be the same. Let it be denoted by *V*. Then, applying Ohm's Law to each arm,

$$(24)\qquad \begin{cases} r_1 i_1 = V_a - V \\ r_2 i_2 = V - V_c \\ r_3 i_3 = V_a - V \\ r_4 i_4 = V - V_c. \end{cases}$$

From the 1st and 3d of these equations,

$$(25)\qquad r_1 i_1 = r_3 i_3;$$

and from the 2d and 4th,

$$(26)\qquad r_2 i_2 = r_4 i_4.$$

Now, since all the current which flows through AB, by supposition, also goes through BC,

$$i_1 = i_2,$$
$$i_3 = i_4.$$

Hence, substituting and dividing,

$$\frac{r_1}{r_2} = \frac{r_3}{r_4}. \qquad (27)$$

One of the common forms which Wheatstone's bridge takes in practice, and known as the slide-wire bridge, is shown in Fig. 275.

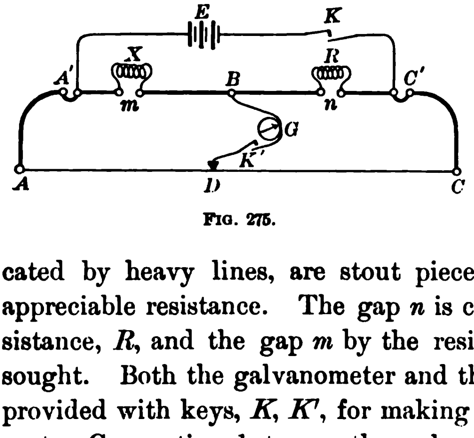


FIG. 275.

From A to C a German-silver wire of uniform cross section is stretched over a scale of equal parts. The portions of the circuit ABC, indicated by heavy lines, are stout pieces of copper, without appreciable resistance. The gap n is closed by a known resistance, R, and the gap m by the resistance whose value is sought. Both the galvanometer and the battery circuits are provided with keys, K, K', for making and breaking the current. Connection between the galvanometer and the wire AC is made by means of the sliding contact D.

In order to obtain a measurement of the resistance X, a position for D is found such that, on closing K and K', there is no deflection of the galvanometer needle.

Under these conditions

$$X = \frac{AD}{DC} R, \qquad (28)$$

by equation 27, since the resistance of the portions AD and DC of a uniform wire may be taken as proportional to their lengths.

393. Post-Office Pattern of Resistance Box. — Fig. 276 shows a form of resistance box which may also be used as a Wheatstone's bridge, the unknown resistance being inserted between A and B. The lettering corresponds to Figs. 274 and 275.

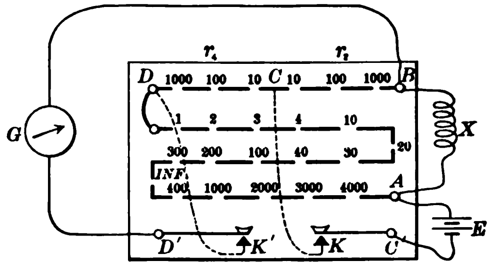


FIG. 276.

394. Bolometer. — The change in the electrical resistance of conductors with the temperature has been applied by Langley to the measurement of the energy radiated by a hot body. This instrument, which he calls the *bolometer*, consists essentially of a Wheatstone's bridge, having two strips of blackened platinum foil, S, T (Fig. 277), inserted in the arms.

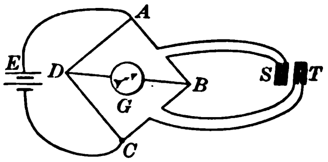


FIG. 277.

If, after a balance has been obtained in the usual way, one of these strips be exposed to waves radiated from the source which it is desired to examine, while the other strip is screened, the change of the resistance, in consequence of the warming of the strip, will cause a deflection of the galvanometer needle. In this way a difference of temperature between the strips of $\frac{1}{1000}$ of a degree may be measured, and one-tenth of this amount may be detected. The bolometer, though not as sensitive as the radio-micrometer, has an advantage in that it may be moved to different positions during an experiment.

EXAMPLES.

1. The end A of a wire ABC is maintained at a potential of 126 volts above the end C. If the resistance of AB is 32 ohms, and that of BC is 19 ohms, what will be the current through the wire, and what the potential difference between A and B?

Ans. 2.48 amperes; 79 volts.

2. If the addition of 5 ohms to a circuit reduces the current to 0.7 of its former value, how many ohms should be added to reduce the circuit to one-fourth of its original value? *Ans.* 35 ohms.

3. The resistance of 10.15 gms. of mercury, when contained in a uniform tube 92.1 cm. long, was found to be 1.059 ohms. What was the specific resistance of the mercury?

Ans. 0.932(10) ohm × cm.

4. What would be the resistance of a copper wire 137 cm. long, and having a diameter of 0.038 mm.? *Ans.* 19.3 ohms.

5. If the resistance of a wire, 0.068 cm. in diameter, is 16.3 ohms, what would be the resistance of a wire of the same length and material having a diameter of 0.024 cm.? *Ans.* 131 ohms.

6. How much German-silver wire, 0.083 cm. in diameter, would be required to make a 6-ohm coil? *Ans.* 1560 cm.

7. What length of iron wire, 0.86 cm. in diameter, will have the same resistance as a piece of copper wire 632 meters long and 0.125 cm. in diameter? *Ans.* 4960 meters.

8. A piece of copper wire 18.12 meters long has a resistance of 3.028 ohms. What would be the length of a coil of the same wire which had a resistance of 22.65 ohms? *Ans.* 135.5 meters.

9. It is found that there is no current through the galvanometer in a Wheatstone's bridge, when contact is made at a point 53.8 cm. from the end of the slide wire which is 100 cm. long. If the resistance in the homologous arm of the bridge is 38 ohms, what is the other resistance? *Ans.* 32.6 ohms.

10. A certain piece of wire has a resistance of 0.082 ohm to the meter. Required the resistance between the angles A and C of a skeleton triangle, ABC, made of this wire, in which AB is 3.7 meters long, BC is 5.2 meters long, and CA is 2.6 meters long. *Ans.* 0.165 ohm.

11. Two wires which separately have resistances of 37.3 ohms and 45.8 ohms are joined in parallel so that a total current of 78.4 amperes flows through the system. What is the resistance of this portion of the circuit, and what current flows through each branch?

Ans. $C_1 = 43.2$ amperes; $C_2 = 35.2$ amperes.

12. A piece of wire having a resistance of 36 ohms is bent into a square and its ends soldered together. The diagonally opposite corners M, N are then connected to the poles of a battery whose resistance is 12 ohms. Compare the current through the battery with that which would pass if M and N were connected by a single straight piece of the same wire. *Ans.* $C_1 = 1.176\ C_2$.

13. A galvanometer having a resistance of 146 ohms is shunted by a coil whose resistance is 65 ohms. If the other resistance in the circuit is 30 ohms, compare the current through the galvanometer with and without the shunt. *Ans.* $C_1 = 0.723\ C_2$.

14. If a galvanometer having a resistance, G, is shunted through a resistance, S, and the other resistance of the circuit be denoted by R, what resistance must be added to keep the main current constant?

Ans. $\dfrac{G^2}{G+S}$.

15. What shunt will be necessary to reduce the sensitiveness of a galvanometer having a resistance of 278 ohms to $\frac{1}{100}$ of its normal value? *Ans.* 2.81 ohms.

16. When a galvanometer having a resistance of 270 ohms is shunted with a 30-ohm coil, it is found that the current through the galvanometer is halved. What was the resistance of the external circuit in the first case? *Ans.* 33.8 ohms.

17. A certain galvanometer having a resistance of 47.8 ohms cannot safely carry a current of more than $\frac{1}{10}$ of an ampere. (*a*) With what resistance should it be shunted to measure a 2.5 ampere current? (*b*) If when so arranged the galvanometer shows a current of 0.084 ampere, what is the current through the circuit?

Ans. (*a*) 2.7 ohms; (*b*) 2.11 amperes.

18. What current must be sent through a tangent galvanometer consisting of 5 turns of wire bent into a circle 45 cm. in diameter, in order that the needle shall be deflected 30° where $H = 0.18$ C. G. S. units?

Ans. 0.74 ampere.

CHAPTER XXIV.

RELATIONS BETWEEN HEAT AND ELECTRICITY.

395. Heating of Conductors. — When a current flows through a conductor, an amount of heat is evolved equal to the diminution of the potential energy of the electricity. Let R be the resistance between two points of a homogeneous conductor having, respectively, the potentials V_1 and V_2, and suppose that a quantity of electricity, Q, passes any cross section in the time t, then, by the definition of potential, the work done by the electricity is

$$W = Q(V_1 - V_2);$$

also, by Ohm's Law,

$$\frac{Q}{t} = i = \frac{V_1 - V_2}{R},$$

whence

$$W = (V_1 - V_2)\, it = i^2 Rt. \tag{1}$$

To reduce this to secondary heat units, it must be divided by the mechanical equivalent of heat. If i is measured in amperes, R in ohms, t in seconds, and H in calories,

$$H = 0.24\, i^2 Rt. \tag{2}$$

Equation 1 was first experimentally verified by Joule, and is often called Joule's Law. Also, as the heating of the conductor, always supposed homogeneous, is independent of the direction of the current, to distinguish it from certain other phenomena observable in a non-homogeneous circuit it is often called the *Joule*, or *irreversible heat effect*.

When the current passing through a wire, stretched horizontally between two points, is gradually increased, the wire will be seen first to sag, then to redden, and, passing through various degrees of luminosity, finally melt. Many useful applications of this heating effect are made in the mechanic arts. Instruments which are likely to be damaged by the passage of too strong a current may be effectively protected by inserting in the circuit a strip of metal which will melt as soon as the current reaches a dangerous amount and break the circuit. Generation of heat by electric currents has many marked advantages over the generation by the use of fuel, in the ease with which the production is controlled, and the directness with which the heat may be applied, as, for instance, in welding, cooking, heating of rooms, blasting, etc.

396. Electric Lighting. — Any body heated to incandescence by the passage of the electric current furnishes an easily controlled source of light. Metallic threads are not available on account of the danger of fusion, whenever the current rises above a definite amount. Carbon has been found to be the most suitable substance for use in incandescent lamps, as it does not pass through the liquid state at ordinary pressures. If it is heated to incandescence in the air, it gradually wastes by oxidation. This result may be avoided by enclosing the thread in a vessel exhausted of air.

The filament of the familiar glow-lamp consists of a strip of bamboo, or other vegetable fiber, carbonized by a special process, and sealed into a globe exhausted to the highest possible degree. The ends of the filament are attached to platinum wires, which pass through the glass and form the electrodes of the lamp. An ordinary 16-candle lamp for use on a 100-volt circuit has a resistance of about 160 ohms

when hot, and takes a current of about 0.6 ampere. It may be said to possess a probable life of 1000 hours of illumination. If a greater current is forced through the filament, the carbon will disintegrate, and the life of the lamp be much shortened. The electric lamp possesses a great superiority over the older method of lighting by combustion, in the fact that there is less heat generated, and no deleterious gases are evolved. Yet even the best electric lamp must be regarded as an extravagant mode of producing light, since 90 per cent of the energy supplied is lost as heat.

397. The Electric Arc. — The flash obtained by the disruptive discharge of a conductor may be made continuous by maintaining two conductors at a constant difference of potential. The light is shown by the spectroscope to be due to the incandescence of particles of the electrodes and of the surrounding gas. The luminous and conducting track may be increased in length by separating the terminals beyond the original striking distance. From its curved form it is called the electric arc. When the arc is formed between two carbon points, they grow very hot and together with the vapor become the source of an exceedingly brilliant light. This result was first obtained by Davy in 1800. The temperature of the arc, estimated by Violle as 3500° C., and by Rosetti as 4800° C., is the highest which can be artificially obtained, and is sufficient to volatilize all substances.

398. The Arc Lamp. — As the carbon points in the electric arc slowly waste away by oxidation, some automatic device to feed them up is necessary, if the light is to be continuously maintained. The mechanism should also be capable of approaching the points within striking distance when the arc is interrupted, and afterward separate them to the proper

working distance. These conditions are attained in practice in a great variety of ways by means of an electromagnetic control.

A typical form of regulator is shown diagrammatically in Fig. 278. The weight of the upper carbon and holder is nearly balanced by the soft iron core *C*, suspended over the wheel *G*, so that, when no current is passing, the point *A* rests on *B*. *D* is a shunt, and *E* a series coil, so arranged that they act differentially on the core *C*. When a current passes through the circuit *HABEK*, the core is drawn into *E*, and the points *A*, *B* are separated a little, after the arc is established. As the resistance increases, a greater portion of the current will pass through *D* until the movable system is in equilibrium with a proper separation of the carbons. By this means the resistance of the arc is maintained practically constant. The carbon pencils in common use are formed of powdered coke made into a paste and baked in the form of rods about a centimeter in diameter.

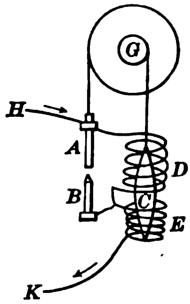


Fig. 278.

In an arc lamp of from 1000 to 2000 candle-power a potential difference of from 40 to 50 volts is necessary. The usual current is from 5 to 10 amperes, and the consumption of the carbons, roughly, 2.5 cm. per hour.

399. Thermo-Electromotive Force. — In 1822 Seebeck discovered that, if one of the junctions of a circuit consisting of two dissimilar metals was maintained at a different temperature from the other, an electric current was produced in the circuit. Thus, if the circuit consist of copper and bismuth, and the junction *H* (Fig. 279) be heated, a continuous current will flow from the bismuth to the copper across the hot

junction, the work done by the current being mechanically equivalent to the heat which is absorbed at the warmer junction. This electromotive force of contact, which is evidently a function of the temperature, may be designated as the *Seebeck effect.*

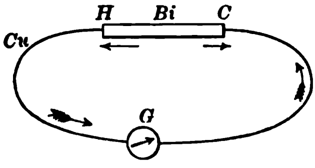


FIG. 279.

400. Peltier Effect.—The converse of the Seebeck phenomenon was discovered by Peltier in 1834, who found that if a current was made to pass the junction of two dissimilar metals, the joint was heated or cooled according to the direction of the current.

Let Fig. 280 represent a circuit consisting of bismuth and copper, of which both junctions are initially at the same temperature. Peltier's discovery is best explained by assuming that there is a small contact electromotive force at the junctions in the direction of the full arrows. If, now, a current be sent through the circuit in the direction of the feathered arrow, work will be done by the current at *W*, and by the electromotive force at *C*. The first junction will accordingly be warmed, and the second cooled.

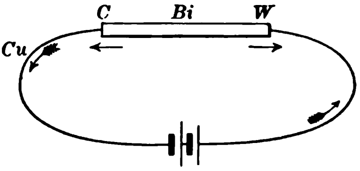


FIG. 280.

The electromotive force at the junction of two metals, as determined by this method, must be carefully distinguished from the difference of potential which is found on joining two metals in air or other gas and then separating them.

The Peltier effect, described in this article, is distinguished from the Joule effect by its reversible character.

The equation for the heating of a portion of a circuit containing two metals may be written

$$H = \frac{Ri^2t}{J} - Pit,$$

where P is a constant called the coefficient of Peltier effect.

It is numerically represented by the heat which is absorbed at the junction when the unit current flows for a unit time.

401. Thermoelectric Series. — If the different metals be arranged in order according to the electromotive force developed per degree difference of temperature between two junctions at ordinary temperatures, the metals will fall in the following series. The adjoined numbers, called *thermoelectric hights*, give the electromotive force in C. G. S. units $\left(\frac{1}{10^8}\text{ volt}\right)$ per degree Centigrade difference of temperature, with respect to lead at 20° C.

It has been shown by experiment that, if two metals are separated in a circuit by several intermediate metals maintained at a common temperature, the E. M. F. is the same as if these metals were joined directly and the junction raised to the given temperature.

THERMOELECTRIC HIGHTS AT 20° C.

Selenium	80700	Gold	120
Tellurium	50200	Tin	10
Antimony . .	2640 to 600	Lead	0
Iron	1750	Mercury	− 41.8
Copper (electrolytic)	380	German silver . .	− 1175
Zinc (pure) . . .	370	Nickel	− 2200
Silver	300	Bismuth .	− 4500 to − 9700

The sign is chosen so that a current running from a higher to a lower metal in the table generates heat at that junction.

402. Thermoelectric Inversion. — It was discovered by Cumming that the order of certain metals in the preceding series was different at high from that at low temperatures. Thus, if one junction in a copper-iron circuit be heated, and the other be kept at 20° C., the current will flow from the copper to the iron at the hot junction, and the electromotive force will increase until the temperature of 284° C. is reached. If, however, the temperature is still increased, the electromotive force will begin to decrease, and finally the current will set in the opposite direction across this junction. This reversal may be obtained more quickly by raising the temperature of the cooler junction. These experiments show that, if both junctions be above a certain temperature, T, the current will flow from the iron to the copper across the hotter junction; that is, in a direction opposite to that which it takes when all the circuit is below T. The temperature T is called the *neutral point*, or *temperature of inversion.*

It follows from the preceding discussion that if one of the junctions be kept at the neutral point, and the other junction be made either hotter or colder, the current will set from the copper to the iron through the junction at the neutral temperature.

403. Thomson Effect. — Since, in the arrangement just mentioned, the neutral junction has no effect, and the E.M.F. of the Peltier effect is in the wrong direction to produce the current, if the second junction is below T, it is evident that there must exist a certain electromotive force in an unequally heated conductor. This conclusion was first reached by Lord Kelvin (Sir William Thomson), who predicted from the laws of thermodynamics, and afterward showed experimentally, that a current from a hot portion to a cold cooled iron but warmed copper. This phenomenon has received the

name of the *Thomson effect.* It is reversible in the same sense as the Peltier effect.

Tait has shown that if τ denote the absolute temperature of the hot junction, τ' that of the cold junction, and τ_n the temperature of the neutral point, the electromotive force of any circuit may be expressed by the equation

$$E = a(\tau - \tau')\{\tau_n - \tfrac{1}{2}(\tau + \tau')\} \tag{3}$$

where a is a constant, depending only on the nature of the two metals composing the circuit.

404. Thermoelectric Diagram. — A valuable method of representing the thermoelectric properties of different substances is to plot the absolute temperatures as abscissas, and the thermoelectric hights, *i.e.* the increase of electromotive force relative to lead per degree rise of temperature, as ordinates. On such a diagram (Fig. 281) the thermoelectric properties of iron will be well represented, for ordinary temperatures, by the line FF', defined by

$$1734 - 4.87\,t,$$

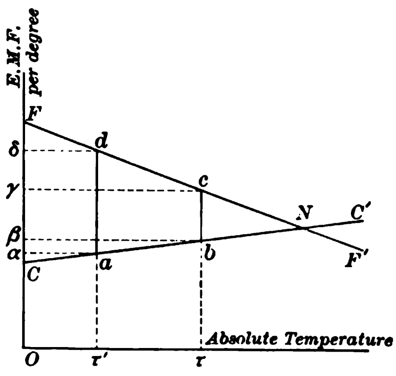


FIG. 281.

t being measured on the Centigrade scale; and copper, by the line CC', given by

$$136 + 0.95\,t.$$

The point N, where these lines intersect, is the neutral point.

To illustrate the use of such a diagram, draw the ordinates corresponding to the abscissas $O\tau$ and $O\tau'$, the absolute

temperatures, respectively, of the cold and hot junctions of a copper-iron pair; then, since thermoelectric hight multiplied by temperature gives electromotive force, the area $bc\gamma\beta$ represents an electromotive force of the Peltier effect. But, as this again is proportional to a quantity of work or heat, the same area may represent the heat absorbed when a current flows from copper to iron through the hot junction. In the same way $da\alpha\delta$ may represent the heat developed, or the Peltier effect, at the cold junction.

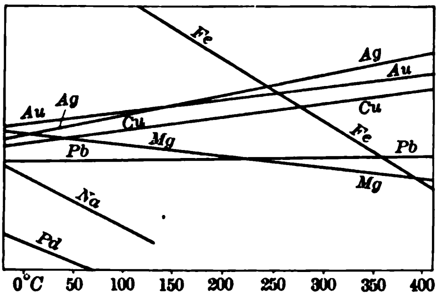


FIG. 282.

It may also be shown that the absorption of heat in the copper by the flow of current from the cold portion to the hot, *i.e.* from a to b, can likewise be represented by the area $ab\beta\alpha$, while that in the iron will be given by $cd\delta\gamma$. Accordingly, the total heat absorbed is proportional to $\alpha abcd\delta$, and that evolved to $da\alpha\delta$. The difference, $abcd$, is the amount transformed into the electrical energy of the current.

In general, the E. M. F. of any thermoelectric chain is given by the area bounded by the ordinates corresponding to the temperatures of the junctions and the lines of the metals in the diagram. The direction of the current is found by traversing the boundary so as always to keep the area on the left. Heat will be absorbed wherever the current sets from a lower to a higher thermoelectric hight, and developed where the current runs from a higher to a lower level. If it is found that the current in one part of the circuit is opposite to that at another, then the areas corresponding

to these opposing electromotive forces must be subtracted, in order to get the resultant electromotive force, which will have its direction determined by the boundary of the larger area.

The diagram from Tait on the preceding page shows the thermoelectric properties of a number of the more important substances.

405. Thermopile. — Although the E. M. F. of a single thermoelectric couple, as shown by the table on p. 443, is extremely small, it is possible, by connecting a number of such pairs in series, as shown in Fig. 283, so that alternate junctions, *p*, *r*, *t*, may be heated, to obtain a resultant electromotive force comparable to that from a voltaic cell. Such an arrangement is known as a *thermopile*. One of its principal applications has been to the detection of small differences of temperature. If a series of thermoelectric couples be arranged in a block so that the radiations from a hot body may fall on the odd junctions, and if the terminals of the pile be connected through a sensitive galvanometer, a very small elevation of temperature at the exposed face of the thermopile will be indicated by the deflection of the galvanometer mirror. By the use of such a piece of apparatus, Melloni made his experiments, now classic, on the diathermancy and emissive powers of various bodies. For the production of currents sufficient for many laboratory purposes, the pile takes the cylindrical form (Fig. 284), which is built up of a number of rings, each containing ten couples made of iron and an antimony-zinc alloy, properly supported and insulated. The interior junctions, which are heated by a central gas jet,

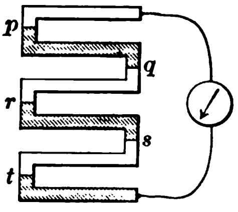


FIG. 283.

are embedded in fire clay to protect them from contact with the flame. A pile of 120 such elements will give an E.M.F.

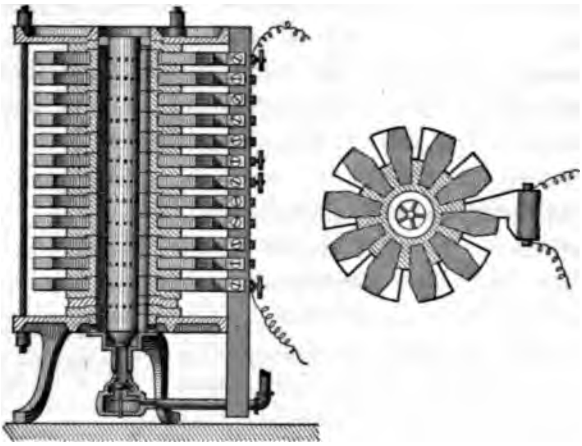


Fig. 284.

of about 8 volts and have a resistance of about 3 ohms. As a current generator the thermopile is but little used, for the E.M.F. is small and unsteady.

406. Radio-Micrometer. — A very sensitive instrument for detecting absorbed radiations, first devised by d'Arsonval, but independently invented and greatly improved by Boys, is the radio-micrometer shown in Fig. 285.

N and *S* are the poles of a strong magnet, between which is suspended a coil consisting of a single loop of copper wire, the purest attainable. To one end of this wire is soldered a small piece of antimony, *A*, and to the other a piece of bismuth, *B*, these in turn being attached to a small projecting strip of copper, *C*. The thermal element is itself surrounded by a tube of soft iron, *H*, in order to shield its diamagnetic material from the influence of the poles. The

radiations from the source which it is proposed to study are allowed to fall on the small strip of copper, where they are absorbed, producing an electromotive force in the thermoelectric pair. Although this E.M.F. is exceedingly minute, the resistance of the circuit is also very small, and a sufficient current is produced to cause a deflection of the mirror. This instrument has been found capable of indicating differences of temperature of the order of one-millionth of a degree Centigrade, and is said to be able to detect the radiations from a candle two miles distant.

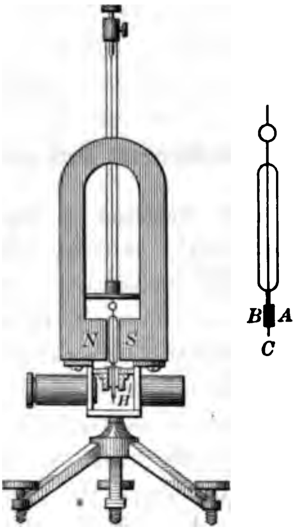


FIG. 285.

In comparison with the bolometer, it has the disadvantage that it cannot be moved during an experiment.

EXAMPLES.

1. How much heat would be generated by a current of 2.32 amperes flowing through a wire whose resistance is 75 ohms for 30 minutes? *Ans.* 175,000 cal.

2. How much will the temperature of 125 gms. of water be raised by a current of 1.39 amperes flowing for half an hour through a coil having a resistance of 5.04 ohms immersed in it? *Ans.* 33.5°.

3. A 16-candle lamp requires a potential difference of 65 volts, and a current of 0.82 ampere. How many watts per candle-power are absorbed by the lamp? *Ans.* 3.33 watts per candle.

4. What power will be required to light 78 incandescent lamps, if the P. D. required for each lamp be 65 volts, and the current 0.83 ampere? *Ans.* 4210 watts.

CHAPTER XXV.

DIMENSIONS AND UNITS OF ELECTRICAL QUANTITIES.

407. Systems of Electrical Units. — The dimensions of electrical quantities take on different forms according as they are defined with or without reference to magnetic phenomena. There have arisen, in consequence, two systems of absolute units, designated respectively as the *electrostatic* and the *electromagnetic* system. Since most of the phenomena of electricity, which are experimentally measured, have to do with their magnetic effects, the second system is relatively of the greater importance. As, however, the electromagnetic C. G. S. units have in practice been found of inconvenient size, they have been replaced for commercial and most scientific purposes by a third or *practical* system.

408. Electrostatic System. — *Quantity.* The electrostatic system of units is founded on the definition of quantity derived from the equation

$$F = \frac{qq'}{Kr^2},$$

setting $[Q]$ for the dimensions of quantity, and observing that $[F] = MLT^{-2}$, and $[r] = L$,

$$Q = [M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-1} K^{\frac{1}{2}}]. \tag{1}$$

As nothing is at present known concerning the dimensions of K, it will be allowed to stand in the formula. The advantage of this retention will be more fully seen hereafter. If in each case the K is divided out, or suppressed, the units so obtained will belong to what is called the electrostatic

system. Thus, from the equation above, the C. G. S. electrostatic unit of quantity is defined as that charge which when concentrated at a point, at a distance of one centimeter from an equal charge, exerts the force of one dyne when the intervening medium is air. This unit may be symbolized by

$$\frac{\text{gm.}^{\frac{1}{2}}\ \text{cm.}^{\frac{3}{2}}}{\text{sec.}}.$$

The electrostatic units have received no special names.

Intensity or Strength of an Electric Field. Electrical intensity is the force per unit charge. Its dimensions are those of force divided by quantity. Thus,

$$(2) \qquad [\mathcal{F}] = \frac{[MLT^{-2}]}{[M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}K^{\frac{1}{2}}]} = [M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}K^{-\frac{1}{2}}].$$

The C. G. S. unit is one dyne per unit charge, or

$$\frac{\text{gm.}^{\frac{1}{2}}}{\text{cm.}^{\frac{1}{2}}\ \text{sec.}}.$$

Potential. Potential is the work per unit charge. Its dimensions are those of work divided by quantity, or

$$(3) \qquad [V] = \frac{[MLT^{-2}]}{[M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}K^{\frac{1}{2}}]} = [M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}K^{-\frac{1}{2}}].$$

The C. G. S. unit is the erg per unit charge, or

$$\frac{\text{gm.}^{\frac{1}{2}}\ \text{cm.}^{\frac{1}{2}}}{\text{sec.}}.$$

Capacity. Capacity is the charge per unit difference of potential. Its dimensions are

$$(4) \qquad [C] = \frac{[M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}K^{\frac{1}{2}}]}{[M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}K^{-\frac{1}{2}}]} = [LK].$$

The C. G. S. unit is 1 cm.

Electric Current. A current in a conductor is the quantity of electricity which passes any cross section per unit of time. Its dimensions are thus

$$(5)\qquad [i]=\frac{[M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}K^{\frac{1}{2}}]}{[T]}=[M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}K^{\frac{1}{2}}].$$

The C. G. S. unit is that current in which the unit quantity passes in a second. It is symbolized by

$$\frac{\text{gm.}^{\frac{1}{2}}\ \text{cm.}^{\frac{3}{2}}}{\text{sec.}^2}.$$

Resistance. By Ohm's Law, resistance is the quotient of potential difference by current. Thus, the dimensions of resistance are

$$(6)\qquad [R]=\frac{[M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}K^{-\frac{1}{2}}]}{[M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}K^{\frac{1}{2}}]}=[L^{-1}TK^{-1}].$$

The C. G. S. unit is $\frac{1\text{ sec.}}{1\text{ cm.}}$, the same formula as for slowness, or the reciprocal of velocity. But no physical significance can be attached to this until some justification is found for the assumption that K is of no dimensions.

409. Electromagnetic System. — *Quantity of Magnetism, or Magnetic Pole.* The electromagnetic system of units is founded upon the definition of the unit quantity of magnetism derived from the equation

$$F=\frac{\mathfrak{m}\,\mathfrak{m}'}{\mu\, r^2}.$$

Substituting the dimensions of force and distance, the dimensions of $\mathfrak{m}$ become

$$(7)\qquad [\mathfrak{m}]=[M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}\mu^{\frac{1}{2}}],$$

the symbol for magnetic permeability being retained as in the case of K, because nothing can be predicated concerning its dimensions.

The electromagnetic system of units, as commonly understood, is that which is found by suppressing the dimensions of μ from the more general formulas.

The C. G. S. unit pole in the electromagnetic system is defined as that pole which, placed at a distance of one centimeter from an equal pole, will exert the force of one dyne, when the intervening medium is air. This unit may be symbolized by

$$\frac{\text{gm.}^{\frac{1}{2}}\ \text{cm.}^{\frac{3}{2}}}{\text{sec.}}.$$

Strength of Magnetic Field. The strength of a field, or the magnetic intensity at a point, is the force per unit pole. Its dimensions are, accordingly,

$$(8)\qquad [\mathbf{H}] = \frac{[MLT^{-2}]}{[M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}\mu^{\frac{1}{2}}]} = [M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}].$$

Magnetic Moment. Magnetic moment is the moment experienced by a magnet divided by the strength of the field. Its dimensions are thus

$$(9)\qquad [\mathbf{M}] = \frac{[ML^2T^{-2}]}{[M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}]} = [M^{\frac{1}{2}}L^{\frac{5}{2}}T^{-1}\mu^{\frac{1}{2}}].$$

Intensity of Magnetization. Intensity of magnetization is the magnetic moment per unit volume. Its dimensions are thus

$$(10)\qquad [\mathbf{I}] = \frac{[M^{\frac{1}{2}}L^{\frac{5}{2}}T^{-1}\mu^{\frac{1}{2}}]}{[L^3]} = M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}\mu^{\frac{1}{2}}.$$

Magnetic Induction. Taking the definition of magnetic induction from

$$\mathbf{B} = \mu\mathbf{H},$$

the dimensions of **B** become

$$[\mathbf{B}] = [M^{\frac{1}{2}} L^{-\frac{1}{2}} T^{-1} \mu^{\frac{1}{2}}]. \tag{11}$$

Current Strength. The fundamental fact of electromagnetism is, that a coil conveying a current, i, and bounding an area, A, in a medium whose permeability is μ, perfectly simulates a magnet whose magnetic moment is

$$\mathbf{M} = \mu i A, \tag{12}$$

from which the dimensions of i are found to be

$$[i] = \frac{[M^{\frac{1}{2}} L^{\frac{5}{2}} T^{-1} \mu^{\frac{1}{2}}]}{[L^2 \mu]} = M^{\frac{1}{2}} L^{\frac{1}{2}} T^{-1} \mu^{-\frac{1}{2}}. \tag{13}$$

The same result may be found from the expression for the magnetic field produced at the center of a circular coil of radius, R, by a current, i, namely,

$$\mathbf{H} = \frac{2\pi i}{R}, \tag{14}$$

whence

$$[i] = [\mathbf{H}L] = [M^{\frac{1}{2}} L^{\frac{1}{2}} T^{-1} \mu^{-\frac{1}{2}}]. \tag{15}$$

The C. G. S. electromagnetic unit of current is that current which, flowing in a conductor one centimeter long (see Fig. 250), would produce a force of one dyne on a unit pole at the distance of one centimeter. It may be symbolized by

$$\frac{\text{gm.}^{\frac{1}{2}}\ \text{cm.}^{\frac{1}{2}}}{\text{sec.}}.$$

Either equation 12 or 14 may be used to find the dimensions of the magnetic quantities in the electrostatic system, in terms of the dimensions of i. These values will be found in the table on page 456.

Quantity of Electricity. The quantity of electricity which passes any section of a conductor is the current multiplied by the time; hence, the dimensions of $[Q]$ are

$$(16) \qquad [Q]=[i]\,[T]=[M^{\frac{1}{2}}L^{\frac{1}{2}}\mu^{-\frac{1}{2}}].$$

Potential Difference, or Electromotive Force. Potential, or the work per unit charge, has the dimensions

$$(17) \qquad [V]=\frac{[ML^2T^{-2}]}{[M^{\frac{1}{2}}L^{\frac{1}{2}}\mu^{-\frac{1}{2}}]}=[M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}\mu^{\frac{1}{2}}].$$

The C. G. S. electromagnetic unit of electromotive force is symbolized by

$$\frac{\text{gm.}^{\frac{1}{2}}\ \text{cm.}^{\frac{3}{2}}}{\text{sec.}^2}.$$

Resistance. By Ohm's Law the dimensions of resistance are

$$(18) \qquad [R]=\frac{[V]}{[i]}=\frac{[M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}\mu^{\frac{1}{2}}]}{[M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}]}=LT^{-1}\mu.$$

The C. G. S. electromagnetic unit of resistance is symbolized by $\frac{\text{cm.}}{\text{sec.}}$ It will be noted that the dimensions of resistance become those of velocity only under the arbitrary suppression of the dimensions of μ.

Capacity. The dimensions of capacity, as determined by the equation $C=\frac{Q}{V}$, are

$$(19) \qquad [C]=\frac{M^{\frac{1}{2}}L^{\frac{1}{2}}\mu^{-\frac{1}{2}}}{M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}\mu^{\frac{1}{2}}}=L^{-1}T^2\mu^{-1}.$$

The C. G. S. electromagnetic unit of capacity is symbolized by $\frac{\text{sec.}^2}{\text{cm.}}$.

For convenience of reference, the preceding results are collected in the following table.

DIMENSIONS OF ELECTRICAL QUANTITIES.

NAME.	SYMBOL.	IN TERMS OF K.	IN TERMS OF μ.
Quantity of electricity	q	$M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}K^{\frac{1}{2}}$	$M^{\frac{1}{2}}L^{\frac{1}{2}}\mu^{-\frac{1}{2}}$
Electric intensity	$\mathfrak{F}$	$M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}K^{-\frac{1}{2}}$	$M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-2}\mu^{\frac{1}{2}}$
Potential; Electromotive force	V; E	$M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}K^{-\frac{1}{2}}$	$M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}\mu^{\frac{1}{2}}$
Capacity	C	$L K$	$L^{-1}T^{2}\mu^{-1}$
Specific inductive capacity	K	K	$L^{-2}T^{2}\mu^{-1}$
Current strength	i	$M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}K^{\frac{1}{2}}$	$M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}$
Resistance	R	$L^{-1}TK^{-1}$	$LT^{-1}\mu^{1}$
Quantity of magnetism, or magnetic pole	$\mathfrak{m}$	$M^{\frac{1}{2}}L^{\frac{1}{2}}K^{-\frac{1}{2}}$	$M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}\mu^{\frac{1}{2}}$
Strength of field	$\mathbf{H}$	$M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-2}K^{\frac{1}{2}}$	$M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}$
Magnetic moment	$\mathbf{M}$	$M^{\frac{1}{2}}L^{\frac{3}{2}}K^{-\frac{1}{2}}$	$M^{\frac{1}{2}}L^{\frac{5}{2}}T^{-1}\mu^{\frac{1}{2}}$
Intensity of magnetization	$\mathbf{I}$	$M^{\frac{1}{2}}L^{-\frac{3}{2}}K^{-\frac{1}{2}}$	$M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}\mu^{\frac{1}{2}}$
Permeability	μ	$L^{-2}T^{2}K^{-1}$	μ
Magnetic induction	$\mathbf{B}$	$M^{\frac{1}{2}}L^{-\frac{3}{2}}K^{-\frac{1}{2}}$	$M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}\mu^{\frac{1}{2}}$

Now, since a quantity of electricity is physically the same thing, whether measured in terms of electrostatic or electrokinetic phenomena, it is evident that its dimensions should be the same in either case. Equating the values of q from the table,

$$\frac{[L]}{[T]}=\frac{1}{\sqrt{[K\mu]}}; \tag{20}$$

that is to say, in order that the two expressions for quantity should be congruent, $K^{-\frac{1}{2}}\mu^{-\frac{1}{2}}$ must have the dimensions of velocity. Also, since all the other magnitudes are derived from q, it is obvious that the first column of dimensions in the table will become congruent with the second by the same substitution.

410. Ratio of the Electrostatic to the Electromagnetic Units. — Let the electrostatic unit of quantity, *i.e.* the value obtained by suppressing K, be denoted by $[Q_e] = [M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}]$, and, similarly, the electromagnetic unit by $[Q_m] = [M^{\frac{1}{2}}L^{\frac{1}{2}}]$. Let q_e be the number which expresses the measure of a definite quantity of electricity in electrostatic units, and q_m the number which expresses the same quantity in electromagnetic units. Then, taking the quotient of these two concrete quantities,

$$\text{(21)} \qquad \frac{q_e[Q_e]}{q_m[Q_m]} = \frac{q_e}{q_m}\frac{[M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}]}{[M^{\frac{1}{2}}L^{\frac{1}{2}}]} = v\frac{[L]}{[T]},$$

the result is seen to have the dimensions of velocity. The value of the ratio $\frac{q_e}{q_m}$, which is usually written v, may be found by first measuring a quantity of electricity in electrostatic and then in electromagnetic units. The value thus determined is, in round numbers,

$$v = 3(10)^{10}\frac{\text{cm.}}{\text{sec.}};$$

that is, quite exactly the velocity of light.

Since the numbers which express the measurement of any physical magnitude are smaller in proportion as the unit is larger,

$$\frac{Q_m}{Q_e} = v, \text{ or } Q_m = vQ_e;$$

that is to say, the number of electrostatic units in one electromagnetic unit is numerically equal to the velocity v.

It is further evident that, since all the other quantities in each system may be made to depend on the definition of the unit charge, this same velocity might be obtained from the ratio of the measurements of any of the other magnitudes on page 456.

Comparing some of the more important quantities, it appears that

$$\frac{i_m}{i_e} = \frac{V_e}{V_m} = \frac{Q_m}{Q_e} = v, \text{ and} \tag{22}$$

$$\frac{C_m}{C_e} = \frac{R_e}{R_m} = v^2. \tag{23}$$

These equations enable one to pass from one system of units to another, and also indicate several other methods of measuring v, which, on trial, have given essentially the same value as that already stated.

411. Electromagnetic Theory of Light. — That the remarkable coincidence between v and the velocity of light is not accidental clearly appears from the work of Maxwell, who has showed that an electromagnetic disturbance would be propagated through vacuous space with the velocity of light. He was thus led to propound the theory that light itself is an electromagnetic phenomenon. In the case where K and μ are not both unity, theory shows that the velocity of propagation should be

$$v = \frac{1}{\sqrt{K\mu}}, \tag{24}$$

and, although experiments on the velocity of light in different media do not show so complete a verification of this relation as in the case of empty space, they nevertheless indicate that

the principal part of the velocity is contained in this formula. Equation 24 may also be derived from the dimensional formulas of Art. 409. Thus, let K, without brackets, be the numerical measure of the specific inductive capacity of the medium, and μ its permeability; then, measuring a given quantity of electricity by both the electrostatic and the electromagnetic definition,

$$q_e K^{\frac{1}{2}}\left[M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}K^{\frac{1}{2}}\right] = q_m \mu^{-\frac{1}{2}}\left[M^{\frac{1}{2}}L^{\frac{1}{2}}\mu^{-\frac{1}{2}}\right], \tag{25}$$

whence

$$\frac{q_e}{q_m}\left[\frac{L}{T}\right] = \frac{1}{\sqrt{K\mu}}\left[K^{-\frac{1}{2}}\mu^{-\frac{1}{2}}\right], \text{ or} \tag{26}$$

$$v = \frac{1}{\sqrt{K\mu}}. \tag{27}$$

412. Practical Units. — It is found that if the measurement of such electrical quantities as are met with in practical work be expressed in C. G. S. units, the numbers are generally either very large or inconveniently small. For example, the electromotive forces of all the ordinary forms of voltaic cells are included between one hundred and two hundred millions of electromagnetic units; so, too, the resistance of a mile of ordinary telegraph wire is not far from ten thousand millions of electromagnetic units. This difficulty has been met by the formation of a new, or *Practical System*, founded upon the electromagnetic system, in which the fundamental units chosen are

$$L = (10)^9 \text{ centimeters},$$
$$M = 10^{-11} \text{ grams},$$
$$T = 1 \text{ second}.$$

The electrical units so derived, with their names, are shown in the following table:

TABLE OF PRACTICAL UNITS.

Magnitude.	Name.	Value in C.G.S. Electro-magnetic Units.	Value in C.G.S. Electro-static Units.
Resistance	Ohm	10^9	$\frac{1}{9}\,10^{-11}$
Potential Difference, or Electromotive Force	Volt	10^8	$\frac{1}{3}\,10^{-2}$
Current	Ampere	10^{-1}	$3(10)^9$
Quantity	Coulomb	10^{-1}	$3(10)^9$
Capacity	Farad	10^{-9}	$9(10)^{11}$
	Microfarad	10^{-15}	$9(10)^5$
Work / Energy = Volt × Coulomb / Ampere × Ohm × Second	Joule, 10^7 ergs.		
Power = Joule per second / Ampere × Ohm	Watt, 10^7 ergs per sec.		

The *joule*, or unit of energy, is the work which would be done in carrying a coulomb between two points differing in potential by one volt. It is also the work which would be done by an ampere in flowing for one second through a resistance of one ohm.

The *watt* is the rate at which work is done by an ampere, flowing through a circuit having the resistance of one ohm. The prefix *mega-* is sometimes used in connection with a physical unit to designate a quantity one million times as great. Similarly, *micro-* signifies one-millionth of the unit to which it is prefixed, while *kilo-* and *milli-* stand, respectively, for one thousand and one-thousandth.

The legal definition of the ohm has been given in Art. 353, the volt in Art. 349, and the ampere in Art. 358.

CHAPTER XXVI.

INDUCTION OF CURRENTS.

413. Faraday's Discovery. — From consideration of the fact that a charged body will electrify another, Faraday was led to investigate whether a conductor, in which a current was flowing, might not also excite, by influence, a state similar to its own in a neighboring conductor. A series of experiments, extending over seven years, yielded nothing till 1831, when he observed that at the instant of making and breaking a current in the vicinity of a circuit containing an unusually sensitive galvanometer, the needle was momentarily disturbed. Although this was not exactly what Faraday was seeking, namely, a permanent change in the state of the galvanometer circuit, it proved to be a discovery of the highest practical importance.

414. Phenomena of Current Induction. — By further examination of the state of a secondary circuit, under various circumstances, Faraday showed that there were several different, though not essentially distinct, ways in which currents could be induced in any closed circuit.

1°. By variation of the primary current.

Let A (Fig. 286) be the primary circuit containing a voltaic battery, and some device by which the current may be made and broken, or have its intensity varied.

Let B be the secondary circuit, containing a galvanometer so placed as to be unaffected by the direct influence of the battery current. It is then found that when the primary current is started, the galvanometer indicates an inverse

current through the parallel portion of the secondary bb', that is, one in a direction opposite to that in the primary aa', but if the strength of the primary current is maintained constant, the induced current immediately disappears. If the primary current be now stopped, an induced current is observed in the secondary, which has the same direction through bb' as the original current through aa'. It will also be found that any increase in the strength of the primary produces an inverse current in the secondary, while a decrease in the primary current produces a direct current in the secondary; but as long as the primary current remains constant

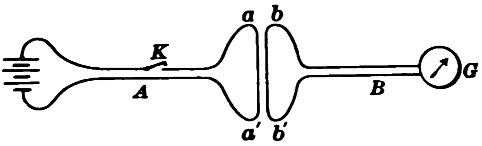


FIG. 286.

there will be no effect in the secondary circuit. The induced currents will be much increased by arranging the two wires in concentric insulated coils, and still more by introducing a soft iron core into the coils.

Entirely similar phenomena of *self-induction* take place within the primary circuit when its current is varied. While the current is increasing there is an induced inverse electromotive force, and while the current is decreasing a direct one. The latter may be readily shown by opening a circuit containing an electromagnet, while the current is flowing, when a bright spark will be formed at the gap. No such effect, however, is observed when the circuit is closed.

It will be noted that in the case of self-induction, just as in the oscillatory discharge of a Leyden jar, electricity behaves as if it possessed momentum whose changes are proportional to the force applied and the time during which it acts.

2°. Induction by motion of the primary with respect to the secondary circuit.

Let the primary and secondary circuits be so arranged as to contain the coils *L* and *M* (Fig. 287) in close proximity. Then, if while the current is flowing in the primary the coils be approached, a brief inverse current will be induced

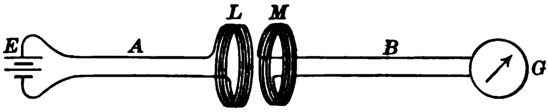


Fig. 287.

in *M*, but if the coils be separated, a direct current will be induced. Also, any variation in the form of the secondary circuit will give rise to similar effects.

3°. Induction by the motion of a magnet with respect to a secondary circuit.

If the coil *L* of Fig. 287 be replaced by a magnet, with its north pole in the same direction as the north face of the coil, the phenomena produced by approaching or withdrawing the magnet from *M* will be the same as those obtained by moving the primary circuit.

415. Laws of Current Induction. — The laws of current induction may be thus stated:

Any change in the magnetic field, with respect to a conductor, induces a current in the conductor whose direction is such as to oppose the change which produced it, and the induced electromotive force is proportional to the rate of change of the field.

This rule for the direction of the induced currents, so far as it relates to those arising from the actual motion of the conductors or of magnets, was first stated by Lenz, and is commonly known as Lenz's Law.

Another statement of the law of induced currents, in the language of Faraday, is, that the induced electromotive force is proportional to the number of lines of force which are cut per second by the conductor. In this connection Fleming has given the following rule for determining the direction of the current. Let the thumb of the right hand (Fig. 288) be pointed in the direction of the displacement, and the index finger in the direction of the lines of force, then the middle finger, set at right angles to the plane of the other two, will give the direction of the current.

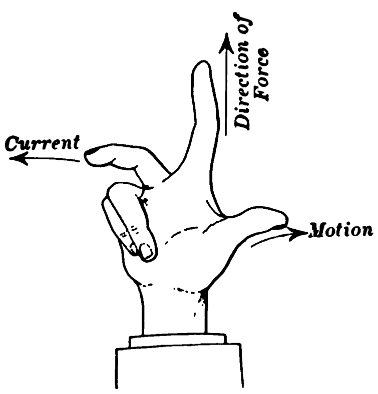


FIG. 288.

Another statement of the law of induced currents, due to Maxwell, is, that the induced electromotive force in any circuit is equal to the rate of decrease of the magnetic induction through the circuit.

The direction of the induced current is here called positive with respect to the field, when the relations are those considered in the right-handed screw rule of Art. 362.

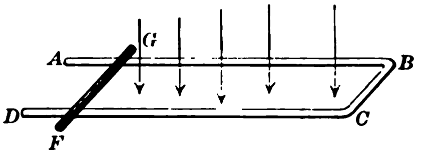


FIG. 289.

416. Application of the Rule. — In illustration of the preceding rules, suppose that a portion of the circuit is a horizontal frame, *ABCD* (Fig. 289), upon which is placed a slider, *FG*, and that the magnetic induction is downward through the circuit. If, now, the slider be moved toward

the end BC, the induction will be diminished, and by Maxwell's rule a current will be induced in the positive direction with respect to the lines of force, *i.e.* $GBCF$.

Again, let PQ (Fig. 290) represent a portion of a primary circuit, and $VSTU$ the secondary circuit. Then, if a current start to flow in the primary from P to Q, the region about this portion of the conductor may be regarded as surrounded by circular lines of force which expand as the current increases. Then, since the induction through the area enclosed by the secondary is increasing, the direction of the induced current will be negative with respect to the lines of force, *i.e.* in the direction $STUV$.

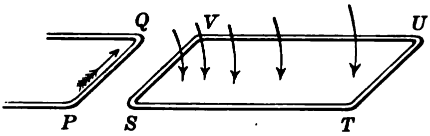


FIG. 290.

Suppose, for instance, that the north end of a magnet is thrust into the coil $PQRO$ (Fig. 291). Then, since by Lenz's Law the induced current must oppose the motion,

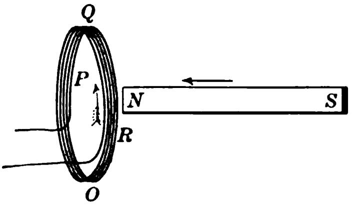


FIG. 291.

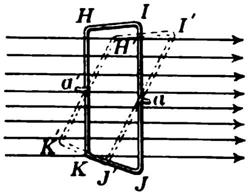


FIG. 292.

the face of the circuit nearest N will be north, or the current will flow in the direction $RQPO$ when the circuit is closed.

As another example, let $HIJK$ (Fig. 292) be a coil which may be made to revolve about a horizontal axis, aa', and suppose that it is placed in a uniform field, so that the lines

of force are perpendicular to the plane of the coil. Call the angle IaI', made by the coil at successive positions with the vertical θ, then, as θ increases from 0 to $\frac{\pi}{2}$, the induction is decreasing, and the induced current will have the direction $HIJK$. From $\frac{\pi}{2}$ to π the magnetic induction through the coil is increasing, but as it has the opposite direction through the circuit the induced current will still be $HIJK$. Further application of the rule will show that from $\theta = \pi$ to 2π the current will have the direction $KJIH$.

If θ is described uniformly with the time, it may be shown that the induced electromotive force E can be written

$$E = A \sin \theta, \tag{1}$$

where A is a constant.

417. Verification of the Law of Induced E.M.F. — The law that the induced electromotive force is proportional to the rate of change of induction may be verified by placing a ballistic galvanometer in the circuit of Fig. 291, when it will be found that the first swing of the needle is the same, whether the magnet is withdrawn from the coil in $\frac{1}{50}$ of a second or $\frac{1}{2}$ a second. Therefore, since the same quantity flows through the galvanometer when the time is $\frac{1}{25}$ as great, the electromotive force must have been 25 times as large.

418. Coefficient of Self-Induction. — The coefficient of self-induction may be defined as the total induction through a circuit per unit of the current which produces it. If N denote the induction and i the current, the coefficient of self-induction $\mathbf{L}$ is given by

$$N = \mathbf{L}i. \tag{2}$$

Since the dimensions of N are those of **B** multiplied by an area, those of self-induction will be

$$[\mathbf{L}] = \frac{[\mathbf{B}L^2]}{[i]} = \frac{[M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}\mu^{\frac{1}{2}}]}{[M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}]} = [L\mu].$$

The C. G. S. electromagnetic unit of self-induction is, accordingly, 1 cm., and the practical unit, called the *henry*, is $(10)^9$ cm.

419. Induction Coil. — Since the total magnetic induction through any circuit is altered by varying the number of turns in the secondary, it is evident that the induced electromotive force resulting from any given change in the primary current may be varied at will. Instruments designed to produce an induced current with an average electromotive force greater or less than that of the primary current are called transformers, or induction coils.

420. Ruhmkorff Coil. — The earlier forms of induction coils were designed to obtain spark discharges at considerable potential differences by use of the current from a battery. Fig. 293 shows diagrammatically the arrangement of parts in a common form of induction coil, often called the Ruhmkorff coil, from one of its later improvers. The primary circuit of stout wire is coiled in one or a very few layers about a core, MN, made of a bundle of soft iron wire to exclude Foucault currents (Art. 423). The secondary circuit is of fine wire often many miles in length, wound outside the primary, and insulated from it with great care.

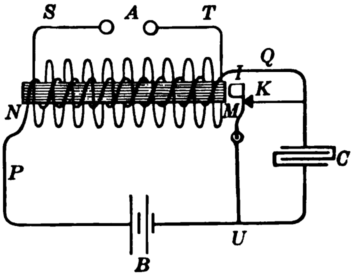


FIG. 293.

To guard against sparking between different layers of the secondary, it is usually wound in a number of disc-like coils, joined in series, but separated by insulating partitions. The primary current flowing from the battery is rendered intermittent by a vibrator, I, which consists essentially of a spring with a mass of iron at the end. When the current magnetizes the soft iron core NM, I is attracted and opens the circuit at K. The spring flies back by its own elasticity,

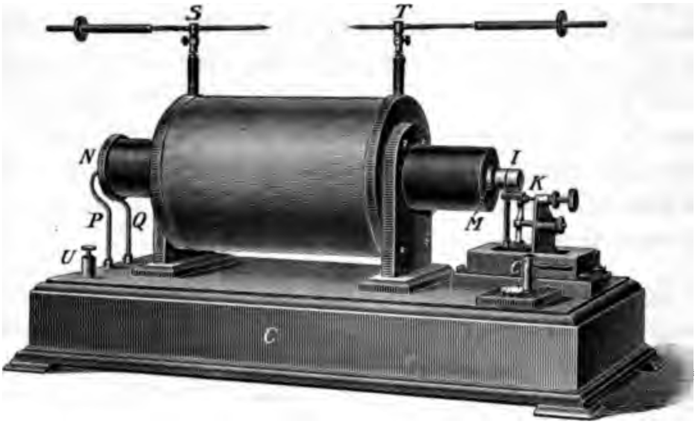


FIG. 294.

closing the circuit, when I is again attracted and the process repeated. If the gap A is not too great, a spark will pass each time the primary circuit is broken. As the self-induction of the primary circuit is considerable, a spark is also likely to form at K, when the current is broken, thus lengthening the time of decay of the field about the secondary. To prevent this, a condenser, C, of large capacity is placed in the primary. Under these circumstances, the induced electromotive force will be either insufficient to form an arc in the primary when the circuit is opened, or

its duration will be greatly shortened. On account of the increased rapidity in the change of the field, the spark at A is considerably lengthened. At the moment when the primary is closed, the self-induction lengthens the time during which the current rises to its full value, and, in consequence, there is in general no spark in the gap A at the "make." A working form of induction coil is shown in Fig. 294, with essentially the same lettering as Fig. 293. The condenser is here enclosed in the base.

The introduction of a condenser into the secondary circuit produces a notable alteration in the character of the spark. The striking distance is smaller, but the brightness of the spark is much increased, and is usually oscillatory in character. When there is no condenser in the circuit, the sparks of short length are usually surrounded with a sheaf of luminous vapor from the electrodes, which is readily blown aside and may be made to ignite a piece of paper.

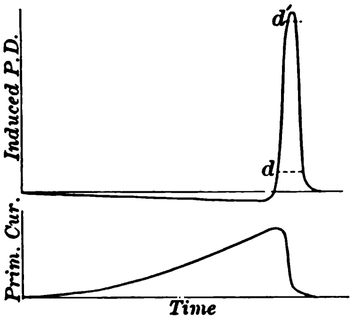


FIG. 295.

A general explanation of these remarkable differences in the phenomena of discharge of a Ruhmkorff coil may be found as follows:

First, let us suppose that, as is ordinarily the case, the terminals have a very small capacity, then Fig. 295 represents the progressive change of P. D. at the terminals, together with the variation of current in the primary coil. If, now, the terminals be brought so near together that a discharge takes place at a moment corresponding to d, we shall at that moment replace the insulating air by the conducting spark track along which we shall have a continuous

flow until the end of the change in the primary. This, a protracted discharge through gases, is the electric arc. But if the terminals are more widely separated, so that the discharge is delayed until a moment represented by d', its duration will be decreased both on account of a delayed beginning and also on account of an earlier termination due to greater resistance between the electrodes. Under the latter circumstances far less of the metals of the electrodes will be vaporized, and the spark will resemble that of the ordinary electrical machine.

Second, imagine that the terminals of the secondary coil possess a very large capacity, such as they would have if connected the one with the inner and the other with the outer coating of a Leyden jar. In this case the P. D. of the terminals would not at all follow the variations of the primary circuit, but would be determined by the magnitude of the charge accumulated by the jar. This process would be more correctly represented by Fig. 296, which shows that, if the separation of the electrodes is a little too great, no spark whatever will occur, but with a suitable approximation we shall have an ordinary Leyden jar discharge in which the secondary coil is in no wise involved, and which is, as has been explained in Art. 303, oscillatory.

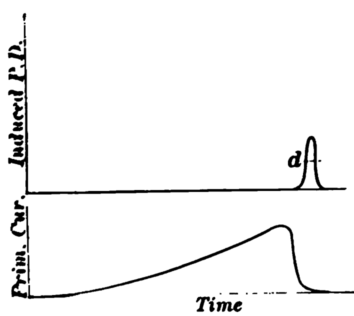


Fig. 296.

421. Tesla Coil. — The oscillations of the electromotive force in the secondary circuit, obtained by mechanical interruption of the primary current, cannot be made to exceed

a few thousand per second. When a greater frequency is desired, the oscillatory discharge of a condenser in the secondary is made to induce a still higher electromotive force in a tertiary circuit. The disposition of parts employed by Tesla, after whom the coil is named, is shown in Fig. 297.

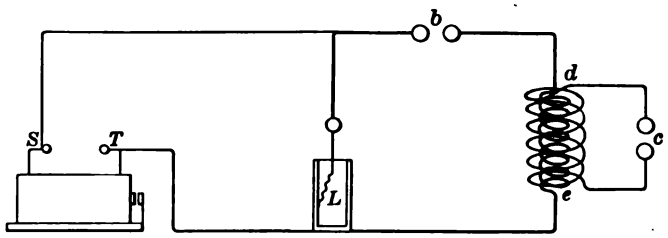


FIG. 297.

The terminals of S, T, of the secondary of a Ruhmkorff coil, are connected, respectively, to the outer and inner coatings of a Leyden jar, leaving an air gap at b so arranged that the discharge shall be oscillatory. The potential differences thus obtained in the third circuit *cde* are of enormous magnitude and a very high frequency. In order to secure proper insulation between the secondary and tertiary circuits, the coils *de* are often immersed in a bath of oil.

The coils used to transform alternating currents, generated by dynamo-electric machines, are discussed in Art. 436.

422. Commutator. — The induction coil is usually supplied with some mechanism for reversing the direction of the primary current through it at will. A form used by Ruhmkorff is shown in Fig. 298.

V is a vulcanite cylinder, to which are attached pieces of copper so as to make a conducting path from A to B and from D to C, through springs which press against the sides.

If the poles of a battery be connected with the points a and d, as in the diagram (Fig. 299), the direction of the cur-

rent through any instrument, *G*, whose electrodes are attached to the springs *B*, *C*, will be in the direction *BGC*. But if the cylinder *V* be turned through an angle of 180°, then the direction of the current through the external circuit will evidently be *CGB*.

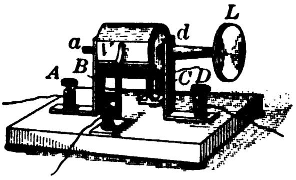


Fig. 298.

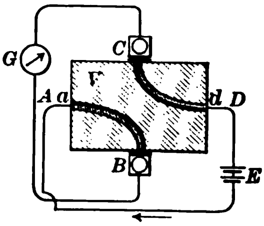


Fig. 299.

Another form of apparatus for securing reversal of a current at will, and known as Pohl's commutator, is shown in Fig. 300. The wires *a*, *d* are fastened to an insulating handle, and dip into mercury cups connected to *A* and *D* respectively. Each of the other mercury cups is connected to the nearest binding screw, and also to the diagonally opposite cup. Suppose that the terminals of a battery are connected to *AD*, and the main circuit to *BC*; then a current entering at *A*, with the commutator in the position shown, would follow the course *AabB* · · · *CcdD*. If the commutator were rocked over, the course would now be *Aab'cC* · · · *Bbc'dD*.

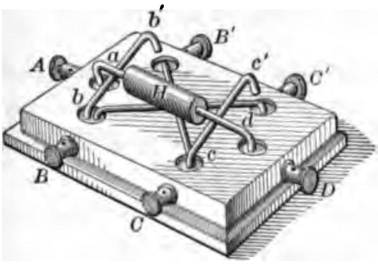


Fig. 300.

423. Arago's Rotation. – Arago discovered in 1824 that, if a copper disc was set revolving beneath a magnetic needle, the needle was deflected in the direction of rotation, and if the motion of

the disc was sufficiently rapid, the needle was carried around with the disc, as if there were frictional connection between them. The phenomenon remained unexplained until the discovery of electromagnetic induction, when Faraday showed that the force exerted by the disc on the needle was due to currents induced in the copper. By Lenz's Law their direction is such as to oppose the motion. Thus, if *N* (Fig. 301) be one pole of the needle, suspended over a disc revolving in the direction of the arrow, eddies, which circulate in the direction shown, will be formed before and behind the pole.

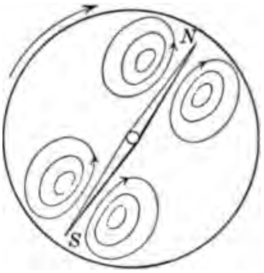


FIG. 301.

Foucault found that a copper disc, rotated between the poles of a strong magnet, soon became hot. In this case a part of the energy expended in revolving the disc is first transformed into the energy of electric currents, and then into heat. The eddy currents just mentioned are sometimes called Foucault currents..

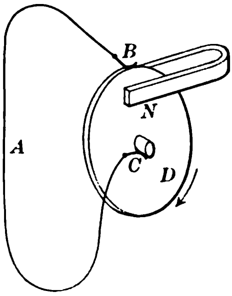


FIG. 302.

424. Faraday's Disc Machine. — Faraday at once followed up his discovery of magneto-electric induction by the invention of several machines which would produce continuous currents. One of the earliest forms, possessing the highest interest, historically, as the germ from which has been developed the powerful magneto-electric machinery of the present day, is shown in Fig. 302. *D* is a copper disc revolving between the poles of a magnet.

The wire of the external circuit is connected to the edge of the disc by a contact spring at *B*, and to the axle by another at *C*. When the disc is revolved as indicated by the arrow, a continuous current is maintained through the wire in the direction *CBA*, as will appear should be the case from consideration of Fig. 301, or an application of Maxwell's rule, Art. 415.

425. Dynamo-Electric Machinery. — Any apparatus designed to produce electrical currents by the expenditure of mechanical work, as distinguished from chemical energy or heat, is called a dynamo-electric machine. In every such machine there may always be distinguished: 1°, the *inductors*, or magnets by which the field is produced; 2°, the *armature*, or conducting system, in which an electromotive force is induced by a change in the field; 3°, the *collectors*, or *brushes*. Machines are called *magnetos* when the field is produced by a permanent magnet, and *dynamos* when it is produced by an electromagnet.

426. Simple Magneto. — The simplest form of electrical machine is a loop of wire having its ends attached to two

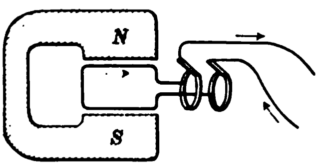


Fig. 303.

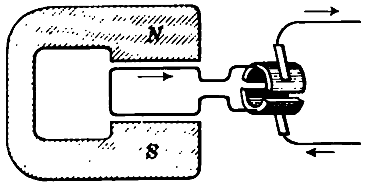


Fig. 304.

rings, and arranged so as to revolve between poles of a magnet (Fig. 303). Resting on each ring is a metallic spring or brush, which serves to connect the external circuit with the armature. If the top wire of the loop be considered as mov-

ing toward the observer, the current at the instant represented will have the direction of the arrows. The direction of the electromotive force reverses every half-revolution, as has been already explained in Art. 416, and the current delivered will be an alternating one. For this reason the machine may be called an *alternator.*

If the ends of the coil be attached to a cleft cylinder (Fig. 304) or commutator, as it is called, the current will be constantly in one direction; but its intensity will vary from zero to a maximum during one half-revolution.

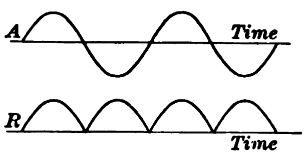


FIG. 305.

Plotting the time as abscissa and the current as ordinate, the action of the apparatus of Fig. 303 would be represented by the curve A (Fig. 305), and that of Fig. 304 by the curve R.

Current generators may be classed, according to the winding of their armatures, as machines of the drum, ring, or disc type.

427. Drum Armature. — The drum type of armature, first introduced by Siemens, is distinguished by the fact that the wire is wound across the ends of a drum-like, soft-iron core. In the earlier forms the armature consisted of one long coil, with its ends attached to the commutator bars, as in Fig. 304; but in the modern forms the armature is made up of a large number of short coils, each attached to its own pair of bars. By this means the strength of the current is made more nearly uniform, as appears in Fig. 306.

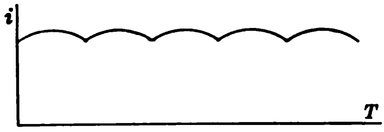


FIG. 306.

428. Ring Armature. — The first form of machine to give a continuous current on a considerable scale was invented by Pacinotti in 1864. The armature consisted essentially of a toothed iron wheel, between the teeth of which the wire was wound in a number of connected coils.

The principle of winding about a soft-iron ring was inde-

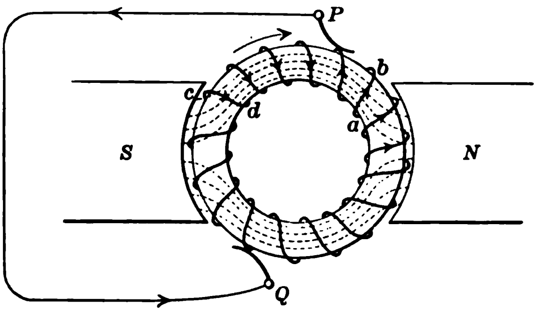


FIG. 307.

pendently invented by Gramme in 1870, whence his name is often associated with this type of armature, which is shown diagrammatically in Fig. 307.

Suppose the ring is revolving in the direction of the arrow, then, applying Maxwell's rule to any single turn of wire, as *ab*, since the induction through this portion of the circuit is decreasing, it is evident that the induced current will run from *a* to *b*, across the face of the ring. Proceeding in the same manner with the others, it is seen that the currents on both sides of the ring are from the bottom toward the top, so that if a brush be placed at *P* and at *Q*, the current in the external circuit will flow from *P* to *Q*. In practice the coil is divided

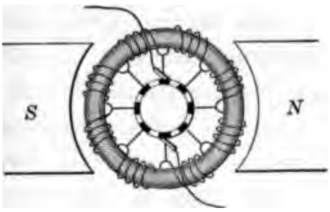


FIG. 308.

into a great number of sections connected in series, and to its own commutator bar, as is shown in Fig. 308.

429. Comparison of Ring and Drum Armatures. — In the case of the ring armature we may regard the total magnetic induction as conducted along two branches, namely, the upper and lower halves of the ring, whereas in the drum armature there is but one branch. Since it requires more wire to surround the two branches than the single one, the cross section of the iron being the same, it follows that the armature resistance is relatively greater in the ring armature. This fact, together with engineering difficulties in the construction of the ring type on a large scale, renders the drum type the prevailing one in powerful machines. On the other hand, in the drum armature neighboring spires may differ considerably in potential, and hence require more careful insulation. An injury to one portion may require the unwinding of the whole, while in the ring armature each section is readily accessible.

The structure of the core of all armatures is laminated, in order to avoid heating from eddy currents.

430. Modes of Excitation. — Dynamo-electric machines may be classified, according to the mode of excitation of the field magnets, as follows:

1°. *Separately excited dynamos*, in which the magnetizing current is supplied by a separate machine, called an exciter.

2°. *Separate-coil dynamos*, in which the magnetizing current is supplied by a separate coil wound on the armature.

3°. *Series dynamos*, in which the whole current passes through the coils of the field magnets, their resistance having been made as small as possible.

4°. *Shunt dynamos*, in which only a portion of the current is allowed to pass through the coils of the field magnets by joining them in parallel with the main circuit.

5°. *Compound dynamos*, in which the field magnets are excited partly by a few turns of wire in series, and partly by a coil connected in parallel with the main circuit.

Each of the preceding methods of excitation has certain advantages depending on the conditions under which the dynamo is required to work.

1° and 2° are used on alternators. 3° will serve where the conditions of the external circuit are unchanging. 4° and 5° are used when, as in a lamp circuit, the conditions are variable.

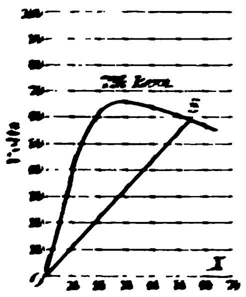


FIG. 309.

431. Characteristic Curves.—A valuable method of studying the behavior of any dynamo is to plot as coördinates the potential difference at the terminals of the machine and the current furnished under various conditions. Thus, the curve of Fig. 309 shows the changes in the action of a series dynamo, run at constant speed, as the external resistance is changed. The resistance at any state, *S*, is given by the tangent of the angle *SOX*. If the external resistance exceed a certain amount, it is evident from the figure that the action of the machine will fall almost to zero. The characteristic of a shunt machine, shown in Fig. 310, has quite a different form. In this case the machine will not work if the external resistance is less than a certain value given by the slope of the portion of the curve passing through the origin. For all values of

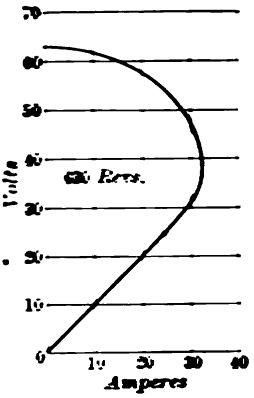


FIG. 310.

the resistance greater than this the current has a definite value.

In order to supply incandescent lamps, which are connected in parallel, it is necessary to sustain a constant potential difference in the mains, whether few lamps or many are on. Since, as is seen in Fig. 309, the characteristic rises as the current increases in the series dynamo, but falls (Fig. 310) in the shunt machine, it is evident that a combination of the two, *i.e.* a compound winding, might be made self-regulating, within certain limits, for a constant voltage circuit. When it is desired to maintain an approximately invariable current, as, for instance, in a circuit containing a number of arc lamps in series, constancy is secured by the use of a regulator which automatically adjusts the strength of the field magnets by changing the resistance of a shunt circuit.

432. Alternators. — The simplest possible form of an alternate current dynamo was shown in Fig. 303. In practice it is desirable that the frequency shall be from 50 to 120 alternations per second, and the potential difference from 1000 to 5000 volts. The first of these requirements is secured by increasing the number of poles and sections of the armature many times, and the second by making the field as intense as possible. One method of construction is shown diagrammatically in Fig. 311, where the armature, consisting of a number of coils in one plane, is made to revolve between the poles of a series of electromagnets placed very near together. This is an example of the disc armature mentioned in Art. 426. A diagram of another type of

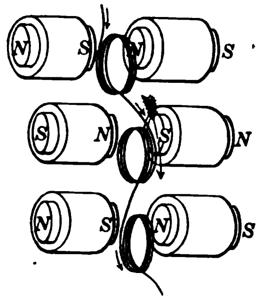


FIG. 311.

alternator is shown in Fig. 312, in which the coils of the armature are wound alternately to the right and left on a ring of soft iron. The core of the armature is sometimes made in the form of a drum, with the coils laid on its periphery. The field magnets and the armature are also frequently made to change places, since better insulation in the armature can be obtained when that remains stationary and the magnets revolve. The field of alternators is usually produced by a small auxiliary dynamo, or *exciter*. The iron cores are in every case laminated to prevent eddy currents.

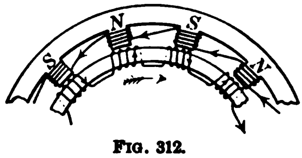


FIG. 312.

433. Measurement of Alternating Currents. — In order to measure the intensity of alternating currents, either the electro-dynamic or the heating effect is used, since each of these may be made independent of the sign of the current. The reading of an instrument constructed on either of these principles, as was shown in Articles 381 and 395, is proportional to the square of the steady current. In case the current is fluctuating, the reading will be that of the mean square of its varying values. If the mean square be denoted by $\overline{i^2}$, the value of the steady current which would give the same reading will be $\sqrt{\overline{i^2}}$, an expression often termed the *virtual current*. The virtual electromotive force is defined in a similar manner as the square root of the mean square of the varying values of the actual electromotive force.

The calculation of these quantities requires a knowledge of the law which the alternations follow. Experiment shows that the E. M. F. and the current may, without serious error, be taken as sine functions of the time.

Thus,

$$E = E_{max} \sin 2\pi nt, \tag{3}$$

where E is the electromotive force at any instant, and E_{max} is its greatest value ; t is the time reckoned from an instant when E is zero, and n is the frequency or reciprocal of the period of an alternation.

Similarly,

$$i = i_{max} \sin (2\pi nt - \phi), \tag{4}$$

where ϕ is a quantity determining the phase of the current.

It is shown in Analysis that the mean value of $\sin^2\theta$ between 0 and 2π is $\frac{1}{2}$, therefore the virtual electromotive force or

$$E_V = \sqrt{\overline{E^2}} = \sqrt{\tfrac{1}{2}}\, E_{max} ; \tag{5}$$

and, similarly,

$$i_V = \sqrt{\overline{i^2}} = \sqrt{\tfrac{1}{2}}\, i_{max}. \tag{6}$$

434. Lag and Lead. — When there is self-induction in the circuit, the current lags behind the impressed E. M. F. This may be represented in a diagram in which the abscissas represent time, the ordinates of the full line (Fig. 313) E. M. F., and the ordinates of the dotted line current strength. It is seen that the current does not reach its maximum till a certain time, $t_2 - t_1$, after the electromotive force has attained its maximum value. This time expressed as a fraction of the period is termed the difference in phase. It appears in the equations as an angle, ϕ, which is the same fraction of a complete revolution that $t_2 - t_1$ is of the period, and is called the lag.

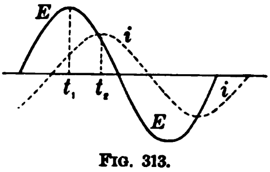


FIG. 313.

The mean power, p_m, or heating effect of an alternating current, may be shown to be

$$p_m = E_V\, i_V \cos \phi. \tag{7}$$

This value evidently vanishes for $\phi = \frac{\pi}{2}$, or a difference of phase of a quarter period. Such a current, in engineering parlance, is called *wattless*.

If the circuit have appreciable electrostatic capacity, ϕ in equation 4 may become positive, and is then called the lead.

435. Virtual Resistance. — Alternating currents do not obey Ohm's Law, but a formal resemblance may be obtained by writing

$$R_V = \frac{E_V}{i_V}, \tag{8}$$

where R_V is called the virtual resistance, or *impedance*. Its value may be shown to be

$$R_V = \sqrt{R^2 + (2\pi n L)^2}, \tag{9}$$

where R is the true resistance, n the frequency, and L the self-induction.

Substituting this value in equation 8, and transforming by the aid of equations 5, 6, and 4, the value of the instantaneous current becomes

$$i = \frac{E_{\max}}{\sqrt{R^2 + (2\pi n L)^2}} \sin\,(2\pi n t - \phi). \tag{10}$$

If the circuit also contain an appreciable electrostatic capacity, C, the value of the impedance must then be written

$$\sqrt{R^2 + \left\{2\pi n L - \frac{1}{2\pi n C}\right\}^2}.$$

436. Transmission of Electrical Energy. — Suppose that a wire which is used to transmit a steady current, i, has a potential, V_1, at one end, and V_2 at the other. Call the

resistance of the circuit R; then, if W be the total energy supplied, Art. 395,

$$\frac{W}{t} = Ri^2 + iV_2, \tag{11}$$

the first term representing the rate at which energy is lost by heating, and the second the rate at which energy is delivered at the further end of the wire.

It thus appears that for a given current the energy delivered will be greater, the larger V_2 is made and the smaller R is made. The great cost of copper will not permit in an extensive system any considerable reduction of the resistance. Successful transmission of power must accordingly be sought in the use of as high voltages as possible.

Alternating currents are most often employed on account of the facility with which they may be transformed and applied to a variety of purposes.

437. Transformers. — The cores of induction coils used to transform alternating currents are always laminated and usually form a closed magnetic circuit. The details of the construction of a common type of transformer are shown in Fig. 314. The core is built up of pieces of sheet iron, about ½ mm. thick, stamped out in the form shown at A. The primary and secondary coils c_1, c_2 are first wound on frames, and the stampings inserted by raising the flaps f, f', as shown at B, care being taken to alternate the joints. Each stamping is insulated from its neighbor by a sheet of paper, and the whole securely bolted together.

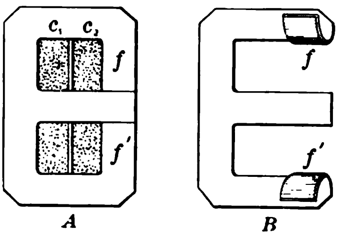


FIG. 314.

438. Motors. — Almost any dynamo may be used to transform the energy of electrical currents into mechanical work. When a current is sent through the field magnets and the armature, the latter experiences a moment which will keep it turning against considerable resistance as long as a current is supplied. Dynamo-electric machines designed to work in this way are called *motors*. Continuous current motors differ so little from the dynamos that a detailed description is unnecessary.

An ordinary alternator, when used as a motor, is at a disadvantage, in that it is not self-starting, and that the field magnets must be separately excited. These difficulties may be overcome by the use of a rotary field, first suggested by Baily in 1879.

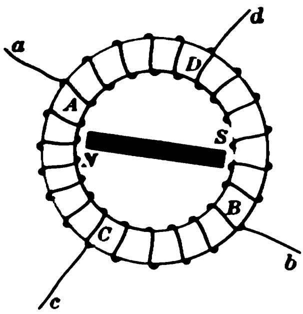


Fig. 315.

One arrangement for producing such a field is indicated in Fig. 315, which represents a coil wrapped continuously about a soft-iron ring. If the terminals of an alternator were connected to this coil by the wires *a* and *b*, the portions of the ring at *A* and at *B* would become alternately north and south poles with every reversal of the current.

Likewise, if *c* and *d* were connected to another alternator, *C* and *D* would be points with similarly changing polarity. If, now, both alternators be worked at the same time, but so arranged that the current through *cd* is a quarter period behind that in *ab*, the poles at *A* and *B* will travel about the ring in the direction *ADBC*. A magnet, *NS*, pivoted at the center of the ring, would be dragged around by the changes in the field, and, similarly, a conductor would be set rotating under the influence of the eddy currents produced.

Rotating fields can be produced by various other combinations of currents, and have been successfully applied to the construction of self-starting motors of high efficiency.

The revolving part of such machines, technically known as a *rotor*, is a sort of wire cage built up on a properly laminated core. It is unconnected with outside circuits, and acquires its magnetic properties solely by the currents which are induced by changes of the field.

439. Motor Dynamos. — By using a motor to run a dynamo, it is evident that from a given current any required current may be obtained. Instead of using two separate machines, the same result may be secured more simply by making the armature with two independent windings. Such an arrangement is called a *motor-dynamo*, or a continuous current transformer if the original and derived currents are continuous. Other machines of this type are called *continuous-alternate* or *alternate-continuous* transformers, according to the nature of the change effected by the machine.

EXAMPLES.

1. Two parallel rods, 135 cm. apart, are joined at one end by a wire and at the other by a sliding rod. When this rod is moved parallel to itself at a velocity of 720 cm. per sec., what electromotive force will be induced in the circuit where the vertical component of the earth's magnetism is 0.453 C. G. S. unit? *Ans.* $4.4(10)^{-4}$ volts.

2. A coil of wire, consisting of 25 turns wound upon a core 40 cm. in diameter, is rotated 30 times a second upon a vertical axis in a field where $H = 0.18 \frac{\text{gm.}^{\frac{1}{2}}}{\text{cm.}^{\frac{1}{2}}\,\text{sec.}}$. What is the mean induced electromotive force? *Ans.* 0.00678 volt.

3. A copper disc, 18 cm. in diameter, is rotated 27 times a second in a field whose intensity is 732 C. G. S. units. What will be the induced E. M. F. between the center and the circumference? *Ans.* 0.0503 volt.

CHAPTER XXVII.

TELEGRAPH AND TELEPHONE.

440. The Needle Telegraph. — The first system of lines for the electrical transmission of signals, on an extensive scale and for commercial purposes, was laid by Wheatstone and Cooke in 1837. The circuits were five in number, each containing a galvanometer needle which might be deflected to the right or left between stops, at will. By agreeing that different combinations of position of the needles should represent the letters of the alphabet, any desired message might be transmitted. It was afterward found that a single needle and circuit was sufficient, since the alphabet could be represented equally well by successive deflections to the right or left. In 1837 also Steinheil showed that a return wire was unnecessary if the extremities of the line were connected to the earth.

441. Morse Instrument. — In 1831 Henry had shown that, if a current was made and broken at one point of a circuit, the attraction and release of the armature of an electromagnet at another part of the circuit would produce a succession of clicks which might be interpreted as signals. Six years later Morse arranged a similar electromagnet so that it would print a succession of dots and dashes, and invented a code which, with a few modifications, has been universally adopted. It was soon found, in practice, that the message could be read with ease from the click of the armature alone, so that the marker is dispensed with, except where an automatic record is desired. When messages are transmitted

over considerable distances, and the currents are very feeble, the receiving instrument, or "sounder," is placed in a local battery circuit which is opened and closed by the lightly pivoted armature of an electromagnet called a *relay* and operated by the current in the line wire.

442. Duplex Telegraphy. — In order to increase the working capacity of a single line, various methods have been devised to effect the simultaneous transmission of several messages.

The duplex system, which permits one message to be sent from each end of the line at the same time, makes use of

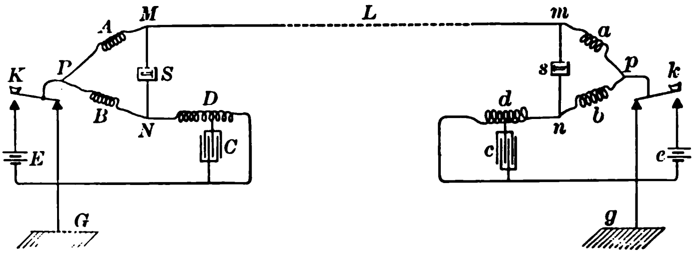


Fig. 316.

the principle of Wheatstone's Bridge. Its working may be understood by the aid of Fig. 316.

Suppose that *L* is the line wire, and let the analogous parts at each end be denoted by large and small letters respectively. *K* is the key, *E* the battery, and *S* the sounder. *A*, *B*, and *D* are three coils whose resistances with that of the line and its opposite end connections form a proportion, and may be regarded as the arms of the bridge. If, now, the key *K* be closed, the current flowing from the battery *E* and dividing at the point *P* will not affect *S*, since, by the arrangement of resistances, *M* and *N* are at the same potential.

The current going through the line wire will divide at *m*, and a portion passing through *s* will operate this sounder. Now, since by the arrangement of the resistances the battery *e* is unable to change the potential difference between *m* and *n*, it is evident that the operation of *K* and *s* is the same whether *k* is open or closed, that is to say, messages may be sent in opposite directions at the same time.

In order to secure satisfactory working in long lines, it is necessary to introduce condensers at *C* and *c*.

443. Diplex System. — A method for sending two messages in the same direction at the same time, invented by

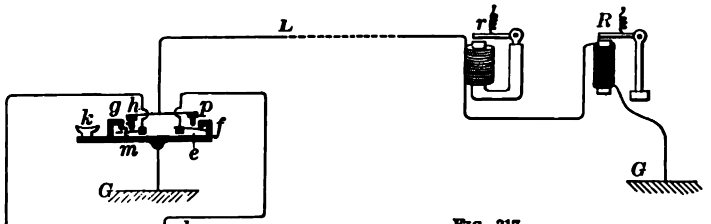
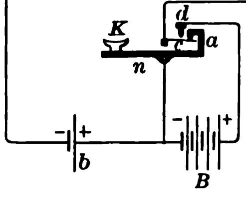


FIG. 317.



Edison and known as the diplex system, is shown in Fig. 317. At *r* is a polarized relay, that is, an instrument containing a permanent magnet so arranged that its armature responds to either a strong or a weak current in one direction, but not at all to a current in the opposite direction. *R* is a second relay sensitive to currents in either direction, provided they exceed a definite value. *B* and *b* are batteries of high and low electromotive force respectively. *K* and *k* are metal keys furnished with projecting fingers at *a*, *f*, and *g*, which may be brought into contact with the springs *c*, *e*, and *i*. If, when the connections are made as

shown, *k* be depressed, the negative pole of *b* will be connected to the earth through *igm*, the positive pole to the line through *nacp*, and *r* will be operated by a feeble positive current. If, on the other hand, *K* alone be depressed, the positive pole of *B* and *b*, which are joined in series, will be grounded through *dcefm*, and the negative pole connected to the line through *i* and *h*. In this case *R* will be affected since the current is strong, but *r* will not respond because the current is in the wrong direction. If *K* and *k* are both depressed, the positive pole of the double battery will be connected to the line through *dcep*, and the negative pole to the earth through *igm*. In this case a strong positive current will operate both relays.

By combining the two preceding systems, it is possible to send two messages simultaneously in each direction. Such an arrangement is called the *quadruplex* system.

444. Multiplex Telegraphy. — If the transmitter send a periodically alternating or intermittent current, and the receiving instrument be so arranged as to respond only to a

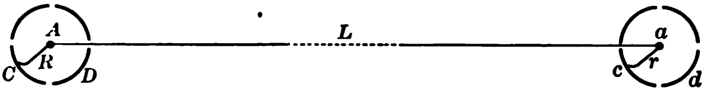


FIG. 318.

current of a special period, then by using instruments of different periods a number of messages may be sent at the same time.

A method for sending several messages at successive intervals of time so short that they are practically simultaneous, was devised by Meyer in 1873.

Suppose that the line wire *L* (Fig. 318) is connected with two arms, *R*, *r*, which revolve rapidly with exact synchro-

nism, making contact with a number of sectors, C, c, etc. Suppose, also, that each of the sectors C, D is connected with a key, and that each of the corresponding sectors c, d is connected with a sounder. Then if the key of C is depressed, the sounder of c, but no other, will respond as often as the arms pass these sectors. The practical difficulties in maintaining perfect synchronism in the distributers R, r are very great, but have finally been overcome.

445. Submarine Telegraph. — The speed at which signals may be sent through an insulated submarine cable is notably affected by the self-induction of the wire and its electrostatic capacity. The wire in this case behaves as the inner coating, the water as the outer coating, and the gutta-percha, or other insulating material, as the dielectric of a condenser. In the process of charging and discharging, a sensible time elapses before the final potential is reached. The retardation of signals in a cable over 600 miles long is too great to permit the use of the instruments ordinarily employed on aerial circuits.

The sounder is usually replaced by a very sensitive galvanometer for a receiving instrument. In the method invented by Varley, and generally adopted, the cable does not form a closed circuit through the galvanometer, but terminates at each end in the coatings of a condenser. If the condenser at one end be charged and discharged, the condenser at the other end will undergo a prompt and similar variation of charge, and the passage of a definite quantity of electricity through the galvanometer is made to deflect the mirror to the right or left, furnishing signals according to a determined code. The speed of working can be very greatly increased by following every momentary current by one of equal duration in a reverse direction. By this means the

signals are transmitted by electrical waves, of which a number may exist simultaneously on a long line. Cable messages may be automatically registered by the ***siphon recorder***, an instrument invented for the purpose by Kelvin. The apparatus consists of a sensitive galvanometer of the d'Arsonval type, in which the suspended coil is made to move one leg of a light glass siphon, suspended by a silk fiber so that the other leg dips into a vessel of ink. By arranging a strip of paper so as to move just below the free end of the siphon, and electrifying either the ink or the paper, the former will be discharged in a fine jet, leaving an irregular sinuous line as the record of the motion of the galvanometer coil.

446. Telephone. — The first electrical instrument for the reproduction of sounds at a distance was devised by Reis in 1861. His transmitter consisted of a diaphragm of stretched membrane carrying an elastic strip of platinum which pressed lightly against the end of a platinum wire, closing a battery circuit. When the membrane of the transmitter was set in vibration by sound-waves falling upon it, partial or complete interruptions of the current were produced by the variable contact of the platinum strip and the end of the wire. The receiver operated on the principle previously discovered by Page, that, when a piece of iron is suddenly magnetized or demagnetized, it will emit a feeble clink. The instrument consisted simply of a piece of iron wire, like a knitting needle, wrapped with many turns of the battery circuit, and mounted on a sounding board. When the coil was traversed by the varying current from the transmitter, the receiver emitted a series of vibrations resembling those experienced by the diaphragm. By this means Reis succeeded, though in a very imperfect way, in transmitting musical sounds, and apparently some words. He also designed another form of

receiver, in which an elastic diaphragm carrying a small armature was set in vibration by the variable attraction of an electromagnet traversed by the current from the transmitter.

In 1876 Elisha Gray devised a transmitter in which the varying immersion in acidulated water of a style fixed to the back of a vibrating diaphragm produced fluctuations of a current without ever actually interrupting them. The varying current was passed through an electromagnet at another part of the circuit, setting in vibration a flexible diaphragm carrying a piece of soft iron.

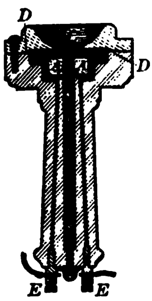


FIG. 319.

The same year Graham Bell invented the telephone which now bears his name, and is shown in section in Fig. 319. *B* is a permanent steel magnet, *A* a coil of wire wound on a spool about the magnet and terminating at the binding screws *E*, *E*, and *D* a thin sheet-iron disc.

When a succession of sound-waves falls on the diaphragm, it is set in vibration, producing a varying field about the coil, and, consequently, induced currents in the circuit. If these currents be made to pass through a similar instrument at some other point of the circuit, they will increase or diminish the attraction of the magnet for the disc. In consequence, the diaphragm is set in vibration and emits a succession of sound-waves almost exactly the counterpart of those which originally fell on the transmitter.

447. Microphone.—In 1878 Hughes made the discovery that, if one portion of a conducting circuit rested lightly against another, very slight vibrations would give rise to fluctuations of the current, without the use of a diaphragm. This principle is illustrated by an instrument called the *microphone*, shown in Fig. 320. *C* is a pointed stick of car-

bon resting lightly in small holes in the two carbon supports *A*, *B*, carried by a light sounding board, *D*. Any slight vibration experienced by *D* will vary the contact resistance at the points of *C*, and give rise to sounds of considerable intensity at the telephone *T*.

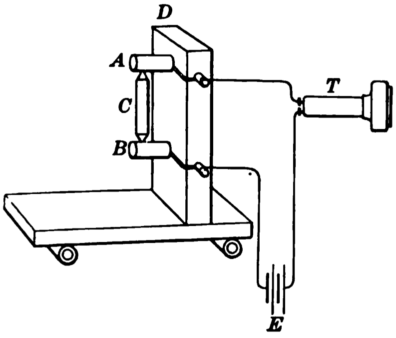


Fig. 320.

448. Transmitter. — In the service of public telephone exchanges, the magneto-electric instrument of Fig. 319 is used solely as a receiver in connection with a transmitter operating on the principle of the microphone.

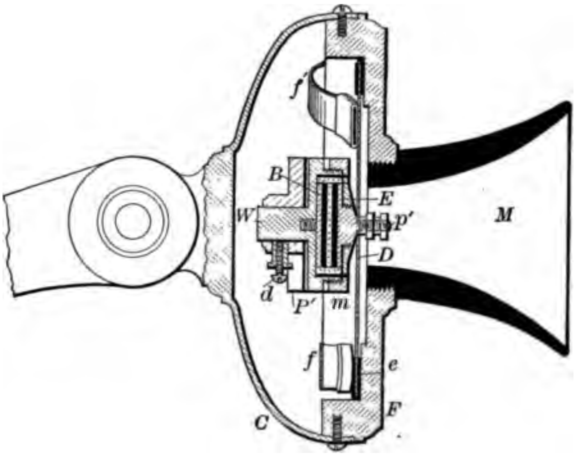


Fig. 321.

In a very successful transmitter, known as the Hunning's type, the fluctuations of the battery current are produced by

passing the current from a diaphragm which is set vibrating by the voice, through a layer of hard carbon granules loosely arranged. Fig. 321 illustrates the form of this instrument used by the Bell Company in their long-distance telephones.

D is the vibrating diaphragm held in place by the springs *f*, *f'*, and isolated from external jars by the rubber packing ring *e*. *B* is a fixed carbon disc backed with metal, and connected to one terminal of a battery. *E* is a similar disc connected to the other terminal, and attached to the diaphragm *D* at *p'*. The space between the discs is filled with granules of hard carbon, whose intimacy of contact varies with the vibrations of *D*, and produces a corresponding fluctuation in the strength of the battery current.

CHAPTER XXVIII.

PASSAGE OF ELECTRICITY THROUGH GASES.

449. Discharge of a Conductor Immersed in a Gas. — The passage of electricity through gases gives rise to an extensive series of complex phenomena. Only a few of the more important features are touched upon in the following articles.

A gas can be regarded as an insulator only at moderate temperatures, even for bodies charged to a low potential. As it is heated, it begins to conduct electricity with a facility depending on the molecular decomposition resulting from the rise of temperature. Experiment indicates that the conduction is convective, and that free atoms are essential to the process. The gaseous molecule does not appear to be capable of receiving a charge. Flames are found to be very good conductors, and exhibit certain properties not unlike electrolytes. For instance, if two dissimilar wires are connected, and the free ends dipped into the flame, there will be an electromotive force around the circuit, amounting in some cases to three or four volts.

The spark discharge of a conductor at low potential in a gas seems also to depend on a fine dust resulting from the decomposition of the electrodes. The amount of disintegration of the kathode is very marked when the spark from an induction coil is passed through an exhausted tube. The glass about the kathode often receives a perceptible metallic film by the deposit of particles torn from the adjacent electrode. The amount of this disintegration depends on the nature of the kathode, but seems to be unaffected by that of

the anode. If ultra-violet light be allowed to fall on a negatively charged conductor, a pronounced leakage occurs, due, apparently, to a disintegration of the surface under the action of the light. The order of sensitiveness of metals to this light effect is roughly that of Volta's Contact Series. No evidence of similar loss of a small positive charge has been observed.

450. Spark Discharge. — The greatest electromotive intensity which a gas can sustain, sometimes called its electric strength, depends not alone on the gas, but also on the material of the electrodes, the state of their surface, their size, shape, and distance apart, and on variations of the electric field, either in time or space.

451. Spark Length and Potential Difference. — The potential difference required to produce sparks between two slightly convex surfaces in air at normal pressure may be expressed as a linear function of the length of the spark.

Baille's experiments lead to the relation

$$V = 4.997 + 99.593\, l \tag{1}$$

for sparks over 2 mm. long, where V is the potential difference in electrostatic units, and l the length of the spark in centimeters.

Experiment indicates that for very short sparks there is a minimum value of potential necessary to produce a spark, and that to produce a spark having a length greater or less than the critical value corresponding to this minimum, a greater potential difference is required.

If the spark length is kept constant and the curvature of the equal electrodes, starting with a plane, is changed, Baille finds that the potential difference increases with the curvature and attains a maximum. This critical value of the cur-

vature decreases with diminishing spark length. The spark potential diminishes with the pressure until the latter reaches a certain critical value depending on the length of the spark, the nature of the gas, the shape and size of the electrodes and that of the vessel containing the gas ; but further diminution of the pressure is accompanied by increase of potential difference.

The potential difference required to produce a spark of given length in hydrogen is much less than in air. Carbon dioxide is stronger than air for short sparks, but weaker for long ones.

452. Discharge without Electrodes. — The phenomena of discharge through rarefied gases are much simplified when metallic electrodes are not used. In this case the discharge may be produced by bringing an exhausted tube into a rapidly alternating field. A convenient method of producing such a field is shown in Fig. 322. The inner coatings of two Leyden jars, *D*, *F*, are connected to the terminals *A*, *B*, of a Holtz machine or an induction coil, and the outer coatings to a wire in which a few turns are made at *C*. If an exhausted bulb be introduced into this coil, when sparks are made to pass between *A* and *B* the rapid oscillations which are set up in the wire produce a sufficient electromotive intensity to cause a bright discharge within the bulb. In order to prevent electrostatic effects in the glass, the coil should be connected to the earth.

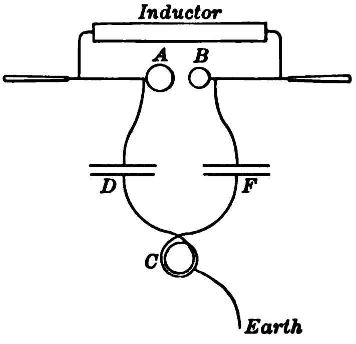


Fig. 322.

When the pressure of the air within the bulb is considerable, no discharge can be obtained; but when it is reduced to about 1 mm. of mercury, a thin thread of reddish light may be observed within the plane of the coil. As the pressure is diminished, the color changes to white, the luminosity increases and attains a maximum, while the discharge appears as a very bright and well-defined ring. As the pressure is still further diminished, the luminosity also decreases; and when a good vacuum has been obtained, no discharge at all passes. The critical pressure in tubes without electrodes is very much less than in tubes in which they are present.

The discharge experiences considerable difficulty in passing across the junction of a rarefied gas and a metal. If, for instance, a metal diaphragm be extended completely across the bulb, the discharge will not cross the metal plate, but breaks up into two separate circuits, as shown in Fig. 323.

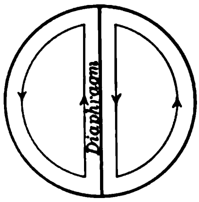


FIG. 323.

The high resistance of a rarefied gas in a tube provided with the usual electrodes has been found to depend chiefly on the difficulty experienced by the discharge in passing from the electrodes into the gas. Experiments on the conductivity of rarefied gases, when the discharge is confined entirely to the gas, show that the molecular conductivity, *i.e.* the specific conductivity divided by the number of molecules per unit volume, is enormously greater than that of the best electrolytes, and higher even than for silver and copper.

The conductivity of rare gases does not obey Ohm's Law, but increases with the electromotive intensity, as might be expected if the discharge is due to the splitting up of the molecules.

453. Effect of Magnet on Electrodeless Discharge. — A magnet deflects the discharge through a tube without electrodes in much the same way as it does a flexible conductor conveying a current. If, for instance, the horizontal bright ring discharge of the preceding article be introduced into a horizontal magnetic field, the ring in those regions where it is at right angles to the lines of force will be divided into two parts, one being raised and the other lowered, for the reason that the discharge is oscillatory. The magnetic field in this case has also the effect of increasing the difficulty of the discharge. If, however, the lines of force are in the direction of the discharge, the effect of the magnet is to facilitate the discharge.

It has been suggested that the streamers in the aurora borealis are intimately connected with this effect of a magnetic field, since the rare air in which the discharges occur is electrically weakest along the lines of force.

454. Discharge between Electrodes in a Rare Gas. — When the discharge passes between electrodes in a tube containing gas at about ½ mm. pressure, the following phenomena may be observed:

1°. A velvety glow, often appearing in patches, runs over the surface of the kathode *k* (Fig. 324). An opaque body placed within this glow casts a shadow on the negative electrode.

2°. From *l* to *b* there is a comparatively dark region, called the *Crookes'*, or *first dark space.* Its length increases as the pressure diminishes, though not in the same proportion. The luminous boundary of this dark space is, roughly, a surface parallel to the surface of the kathode.

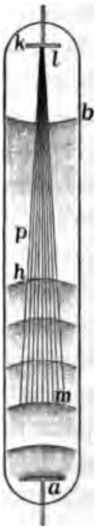


Fig. 324.

3°. Next to the dark space comes a luminous region, *bp*, called the *negative glow*, or *column*, of variable length, but entirely independent of the position of the anode. The negative glow is stopped by any substance against which it strikes, and is checked by too great a restriction of the space between the walls of the tube. The effect of a magnet on the negative column may be described as that which would be observed if it consisted of a paramagnetic substance without weight and having perfect freedom of motion.

4°. Following the negative glow is the *Faraday*, or *second negative dark space*, very variable in length and sometimes wanting.

5°. The remainder of the tube, quite up to the anode, is occupied by a luminous space, called the *positive column*. This region often exhibits striking periodic variations in its luminosity, known as *striae*. The distance between the bright parts increases with the diameter of the tube, provided the striations reach its sides. This distance also increases as the density of the gas diminishes. The striae often have an irregular motion of translation which obscures their distinctness. When observed in a revolving mirror, many discharges appear striated which seem continuous when examined by direct vision.

It is not improbable that the striations exist at pressures much greater than those at which the striae are usually observed. The positive column always takes the shortest path through the gas to the negative electrode, and is regarded as made up of a succession of discharges by which the electricity passes between the electrodes. Its length increases with the length of the tube. The negative dark space and the negative glow are merely local effects depending on the circumstances of the passage of the electricity from the gas to the kathode. By the use of a revolving mirror and a

long discharge tube, J. J. Thomson has measured the velocity with which the flash travels through the positive column, and finds that in air at ½ mm. pressure, in a tube 5 mm. in diameter, the velocity of discharge is rather more than half the velocity of light, the luminosity always traveling from the positive to the negative electrode. The positive column is deflected by a magnet in the same way as a perfectly flexible wire would be when carrying a current in the direction of the discharge through the tube.

6°. Negative rays or molecular streams.

In a highly exhausted tube the luminous effects are chiefly confined to the glass, as if gaseous particles were projected

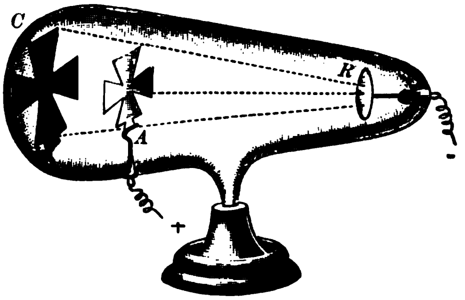


Fig. 325.

at right angles from the kathode, and had the power of exciting luminescence in any substance which will phosphoresce in ultra-violet light.

If either a conducting or insulating screen be placed between the electrodes and the walls of the tube, a shadow is thrown on the glass, that is to say, the phosphorescence is stopped at all points behind the screen (Fig. 325).

The light emitted from any phosphorescing substance within the tube shows a band spectrum characteristic of the

substance. Crookes has further shown that, by using a portion of a spherical shell for a kathode, the negative rays may be concentrated on a platinum wire which is made to glow.

When the rays are allowed to fall on vanes, such as are used in the radiometer, these are made to revolve as if bombarded by particles from the negative electrode. The kathode ray is deflected in a magnetic field in the same way as it would be if it consisted of a stream of negatively charged particles moving away from the kathode. Two pencils of these rays exert a mutual repulsion which is consistent with the same hypothesis.

455. Kathode Photography. — Hertz discovered that the glass of a Crookes tube would phosphoresce when covered with thin flakes of gold leaf, and afterward showed that these metal flakes were more transparent to the kathode radiations than sheets of mica of equal thickness. In 1894 Lenard, by using an aluminum window, succeeded in bringing the kathode rays out into the air, where their presence was detected by the blackening of a photographic plate and the production of luminescence. He also found that dense bodies were rather more opaque than rare ones.

These radiations passed through hydrogen more readily than through oxygen at the same pressure; but if the hydrogen was compressed to the same density, it was as opaque as the oxygen. He also obtained a shadow photograph on a sensitive plate enclosed within a box with an aluminum front. In 1896 Röntgen found that high-vacua tubes which had been covered with black paper produced luminescence in phosphorescent substances, and that bodies differed widely in their ability to transmit these radiations. He found, for instance, the flesh of the hand so much more transparent than the bones that the latter produced a visible shadow on

a fluorescent screen. He also made photographs of objects contained in an aluminum box, and of the bones in living subjects.

The Röntgen radiations seem to differ from those investigated by Lenard, in the distance from the source at which they are perceptible, and by their not being deflected by a magnet.

456. The Arc Discharge. — The arc discharge is characterized by the incandescence of both terminals, the passage of a considerable current, and a comparatively small potential difference. This potential difference is virtually independent of the current, and may be expressed in terms of the length of the arc l, by the equation

$$V = a + bl, \tag{2}$$

where a and b are constants. When V is measured in volts, and l in centimeters, a and b are, roughly, numbers of the order of 30 and 50 for carbon, provided l is not very small.

The great fall of potential occurs close to the anode. It seems probable that the term a (equation 2) is connected with the work required to disintegrate the electrodes.

This disintegration is a very marked feature of the arc discharge, and is not confined to the kathode, as in the case of the passage of the electricity through an exhausted tube. In the case of carbon the loss of the anode is considerably greater than that of the kathode. When the current is continued for some time, the former is hollowed into a cup-like form, while the kathode takes a pointed shape.

457. Theory of the Electric Discharge.—The phenomena of electric discharge are best explained by the view that the passage of electricity through air or through any other gas,

as well as through an electrolyte, is effected by chemical changes. Chemical decomposition on this view, then, is not an accidental accompaniment of the discharge, but the essential feature without which it could not occur. The nature of these chemical changes is most clearly represented by the idea, introduced by Grotthus, of a chain of gaseous particles alternately charged positively and negatively, and oriented by the electrical forces of the field. Suppose, for instance, that P and N (Fig. 326) are two electrodes, and that ab, cd, ef · · · are a series of molecules in such a chain. It may be shown by calculation that it is unlikely that any separation will take place from the unaided agency of the external field. But if the bond between two atoms is nearly broken by a collision, the forces of the field may complete the separation.

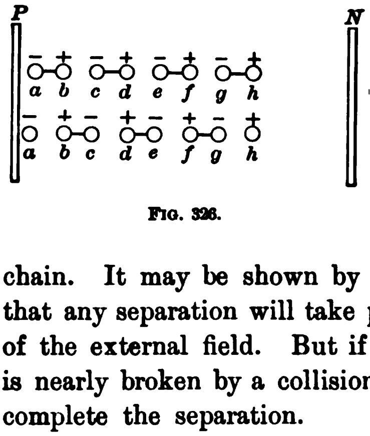


Fig. 326.

Then, if a is freed, it will go to P, b will join to c, d to e, and so on, freeing h, which may serve as a quasi-electrode, from which a new series of recombinations in a consecutive chain originates. Regarding these recombinations as effected simultaneously throughout the chain, it is evident that the discharge will go from one end of the chain to the other in the time that b moves up to c. The velocity of the discharge, as has been seen in Art. 454, is enormously greater than that which can be possessed by the molecules of the gas.

The whole discharge will, then, on this view, consist of a series of non-contemporaneous discharges, which travel consecutively from one chain to the next. This theory is strongly supported by the fact that, in a magnetic field, each stria is subjected to such a deformation as would be the case

if it were a terminated flexible current, starting from the bright head of one stria and ending in the hazy inner surface of the stria in front. Thus each stria is to be regarded as a bundle of parallel Grotthus chains, the bright parts of the stria corresponding to the ends of the chain, and the dull part to the middle.

The fact that the striae are visible only within narrow limits of pressure does not forbid the assumption that they are always present, since it is evident that the gas must possess great regularity of structure to permit them to be seen at all. In the body of the gas, the ions set free at the edge of a stria are close to those of opposite sign, but in the stria next to the electrode, the ions are set free against a metallic surface, which may account for the exceptional character of the stria at the kathode. The chemical changes and the fall of potential at this point are certainly abnormal.

The exact function of the negative rays in the discharge is doubtful. They appear at most to be a local effect, and play but a small part in carrying the current through the gas.

CHAPTER XXIX.

ELECTRIC WAVES.

458. Period of Oscillatory Discharge. — It was shown in Art. 303 that the discharge of a condenser may be oscillatory. The conditions that the oscillations shall be of perceptible magnitude are:

1°. That the rupture of the dielectric shall be very sudden. If the surfaces of the conductors at the air gap are pointed, or even roughened, so that the electricity may escape gradually by the glow or brush discharge, there will be hardly any oscillations set up. An analogous behavior may be observed in the string of a musical instrument. If such a string be displaced sidewise and then suddenly released, it will perform a series of vibrations which produce the sensation of sound. If, however, the constraint of the string be gradually released, no such vibrations will be set up.

It is found necessary, to insure oscillatory discharge, that the discharging knobs of a condenser shall be highly polished.

2°. That the self-induction shall bear a certain relation to the capacity and the resistance. If R be the resistance, L the self-induction, and C the capacity of the circuit, the discharge will be oscillatory when

$$R^2 < \frac{4L}{C}. \tag{1}$$

Its frequency will be

$$n = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}, \tag{2}$$

or, if R is small, the period may be taken for a sufficient approximation, as

$$P = 2\pi \sqrt{LC}.$$

459. Electric Waves. — The phenomena of the oscillatory discharge of a condenser are by no means confined to what goes on within the circuit. With every change of the charge and current within the condenser there is produced a simultaneous change in the electrostatic and electromagnetic field, which spreads just as a disturbance would in any elastic medium. Accordingly, a periodic disturbance, such as the oscillatory discharge, must give rise to a system of waves. That is to say, if a particular condition of things be observed at one instant at a point, then by moving away from that point a certain distance the same condition will be again found; or, after the lapse of a certain time, namely, the period of one oscillation, the condition of things first noted at any point will again be observed at that point.

It was first shown by Maxwell, from theoretical considerations alone, that the velocity of propagation of such a disturbance should be

$$v = \frac{1}{\sqrt{K\mu}},$$

and that in air it was the same as the velocity of light.

Hertz later succeeded in showing the existence of these waves, and in reducing them to the convenient length of a few meters, by using electrical systems having small capacities and small coefficients of self-induction.

460. Oscillator. — One of the forms of apparatus used by Hertz for exciting the electrical oscillations is shown in Fig. 327. A and B are two square zinc plates, to which are

fastened the rods F, G, terminated in the metal balls D and E. The electrical vibrations are set up in this system by connecting F and G with the terminals of an induction coil, which does not, however, sensibly affect the period given by equation 2. In order that the spark which passes across the air gap DE shall be oscillatory, it is necessary to keep the balls smooth and well polished.

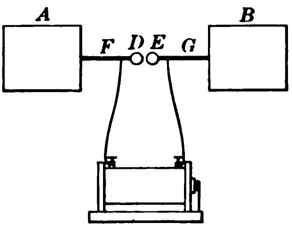


FIG. 327.

461. Resonator. — To detect the presence of waves set up in the space about the oscillator, Hertz made use of an instrument called a *resonator*, which consisted essentially of a circular piece of wire terminated by two balls, forming a spark gap whose width might be adjusted by a fine-pitched screw. When the resonator was brought into the field of the oscillator, the following phenomena were observed.

Case 1. Let the axis FG of the vibrator be placed horizontal, and the line XY (Fig. 328), drawn horizontally through the center of the spark gap DE at right angles to the axis, be called the *base line*. If the resonator be placed so that the base line is perpendicular to its plane at its center, then sparks of maximum brightness will pass when the gap HI is at the bottom or top. If the resonator be turned in its own plane till the spark gap is the horizontal plane through XY, *i.e.* till the line joining HI is perpendicular to the axis of the oscillator, the sparks cease entirely. In all the intermediate positions the spark will have a brightness determined by the location of the gap.

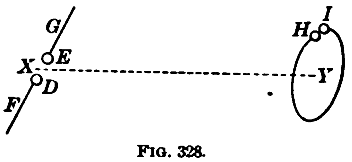


FIG. 328.

Case 2. When the center of the resonator is on the base line, and its plane perpendicular to the axis of the vibrator, no sparks will pass in any position of the air gap.

Case 3. When the center of the resonator is on the base line, and its plane horizontal, sparks pass in all positions of the air gap, but they are longest when the air space is nearest the oscillator, and shortest when farthest from the oscillator.

An explanation of these phenomena may be obtained from a consideration of the electrostatic induction and the induced currents. Case 1 is an instance of electrostatic induction alone. Since for a rapidly alternating current the impedance of the resonator circuit is very great, it is permitted to consider those portions of the resonator near the air gap as two insulated conductors which are placed in the electrostatic field produced for an instant by the charged plates of the vibrator. Under these circumstances, when the air gap is parallel to the axis of the vibrator, the adjacent faces across the air gap of the resonator will take one a positive and the other a negative charge, and if the fall of potential is great enough a spark may pass. If, however, the gap is perpendicular to the axis of the vibrator, it is evident there can be no such inductive action across the gap. Also, as no magnetic lines of force cut the resonator, there will be no sparks from this cause.

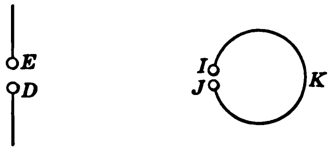


Fig. 329.

In Case 2, since the spark gap is always at right angles to the electrostatic field at that point, and the lines of magnetic force are parallel to the plane of the resonator, there is no sparking in any position.

In Case 3, if the gap be toward the vibrator, then at the instant when D is at its maximum potential J will be at a higher potential than I (Fig. 329). If, now, the insulation at

DE breaks down, the potential of *I* will rise and *J* will fall, and a spark may be sent from *I* to *J* by the quasi-momentum of the electricity. Also, the discharge current from *D* to *E* sets up a magnetic field consisting of circles about and perpendicular to *DE*, which grow in diameter as long as the flow is increasing. These lines of force, cutting across the resonator, give an electromotive force in the direction *JKI*, which conspires with the electrostatic effect to make the spark greatest when the gap is toward the vibrator. When, however, the air space is farthest from the vibrator, these two effects oppose each other, and the spark is a minimum.

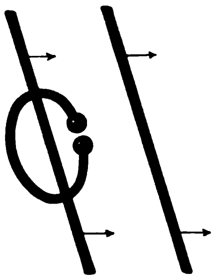


Fig. 330.

The same results may be obtained in a different way by supposing that the vibrator emits a series of closed Faraday tubes which move parallel to the base line with their greater axis parallel to the vibrator gap. The tendency of the resonator to spark will be proportional to the number of these tubes which are caught and stretched across the air gap (Fig. 330).

In Case 1 the sparking will be due only to those tubes which fall directly on the air gap, all others breaking and joining as soon as they have passed the resonator. The sparks can evidently be formed only when the gap is parallel to the axis of the vibrator.

In Case 2 the air gap is always at right angles to the length of the tubes, so that none of them will stretch across the gap.

In Case 3 the tubes will be thrown into the gap whatever its position.

If the gap is in the position nearest to the oscillator, the tubes striking the resonator will break and stretch directly

across. If the space is farthest away, the tube breaks (Fig. 331), each end traversing opposite sides till they reach the gap, where the ends bend together again, and, breaking off, leave a piece in the gap. But since this portion is, by the momentum of the tube, bent more from the shortest line joining the terminals, than when the air space is nearest the vibrator, the field will not be so intense and the spark will be feebler.

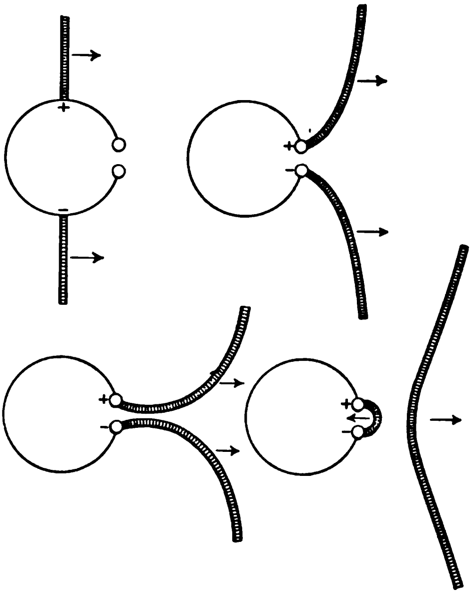


FIG. 331.

462. Resonance. — If the period of free vibration of the secondary circuit, regarded as an electrical system, be variable, the latter may be arranged so that it will give sparks only when its period coincides with that of the vibrator. This phenomenon, called *resonance*, is quite analogous to the case in which one vibrating tuning fork sets another of the same pitch in vibration, but produces no appreciable effect on forks having a slightly different period.

Electrical resonance may be illustrated by the following experiment, due to Lodge. Two similar Leyden jars are placed in circuits of nearly the same size (Fig. 332).

The circuit of A is furnished with a gap, but that of B is closed by a slider, S, by which its self-induction and, conse-

quently, its period may be changed. The inner and outer coatings of this jar are also brought nearly into contact by a strip of tin foil, C, bent over the lip of the jar. The jars having been placed side by side, with the plane of each circuit perpendicular to the line joining their centers, A is charged from an electrical machine. When an oscillatory discharge occurs in the circuit of A, electrical oscillations are set up in B. It is now possible by moving S to find a position where these oscillations are sufficiently violent to produce an overflow spark at C. A slight movement of the slider from this position will destroy the synchronism and stop the sparking.

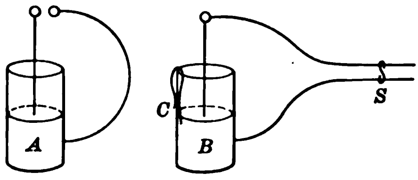


FIG. 332.

463. Other Properties of Electric Waves. — Experiments of Hertz and other investigators have shown that electromagnetic waves may be reflected, refracted, and polarized in exactly the same way as those of ordinary light. In treating of the direction of vibration in an electromagnetic wave it is necessary to observe that there are two sorts of disturbances which appear in the phenomenon — the *electric*, which is parallel to the axis of the oscillator, and the *magnetic*, which is at right angles to this axis, the direction of propagation being perpendicular to both of these disturbances. In a beam of plane polarized light the electric displacement is perpendicular to the plane of incidence.

The oscillations obtained in the Hertz vibrator, or in the Leyden jar, having a frequency, say from one to a thousand million per second, radiate their energy so rapidly that they entirely cease after three or four, or at most a dozen, vibra-

tions. These oscillations, however, are slow compared to those which are required to excite the retina. For example, the wave-length of the longest visible waves in the red is about $0.76(10)^{-4}$ cm., and the velocity of light is $3.(10)^{10}$ cm./sec., whence the frequency must be about $4(10)^{14}$ per sec.

Since the energy of an electrical oscillation is rapidly dissipated in any conductor, it follows, as a consequence of the electromagnetic theory of light, that all conductors must be opaque.

464. Waves in Wires. — If the potential at one end of a wire be varied periodically by the discharge of a condenser, the potential at successive points will vary as a periodic function of the distance of the point from that end of the wire. The only effects produced in the wire are confined to an extremely thin skin at the surface, so that the velocity of propagation of the disturbance remains the same as for air. That the velocity of the wave is independent of the material of the wire which guides it, may be illustrated as follows: Let *ABCD* (Fig. 333) be a rectangle of copper wire with an air gap at *EF*. Then, if a pulse from the discharge of a Leyden jar be sent along the wire *JK*, it will divide at *K*, and each portion reach *EF* at the same instant, provided *K* is at the center of *AB*. Under these circumstances there will be no spark across the gap. If, however, the point *K* be shifted to the right or left, so as to make a difference of path to the gap, sparks will appear. Now, if while *K* is at the middle of *AB* one side of the copper rectangle be replaced by an iron wire, no change of the node at *EF* will be observed.

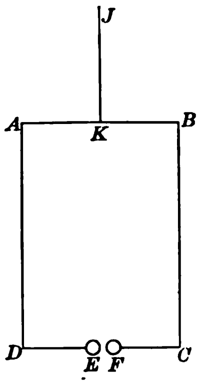


Fig. 333.

PART IV. — SOUND.

CHAPTER XXX.

WAVES.

465. Definition of a Wave. — A wave may be defined as a progressive form due to the periodic vibration of the particles of the medium through which it moves.

When the direction of vibration is perpendicular to the direction of propagation, the wave is termed a *transverse* wave. When the direction of vibration is parallel to that of propagation, the wave is called a *longitudinal* or *compressural* wave.

466. Transverse Waves. — Suppose that a series of particles, originally in a horizontal straight line, be given a vertical displacement, according to the harmonic law,

$$y = a \sin bx, \tag{1}$$

y being the displacement, x the abscissa of any point in the row, and a and b constants. Then the curve P (Fig. 334) will represent the position of these particles at any instant in a medium which is transmitting a transverse wave. The point C, where the curvature is the greatest, is called the crest; and T, where it is least, because negative, is called the trough. If the wave is traveling toward the right, then an instant later the displacement of C will have diminished, and that of m will have increased to its maximum, the crest in the meanwhile having moved from C to m.

The distance from crest to crest, or, more generally, between the nearest points which are in the same phase, is called the wave-length. Since while each particle performs a complete vibration the crest will advance a wavelength, the general expression for the velocity of a wave will be

$$v = \frac{\lambda}{P} = n\lambda, \tag{2}$$

where λ stands for the wave-length, P for the period, and n for the frequency or the number of vibrations per second.

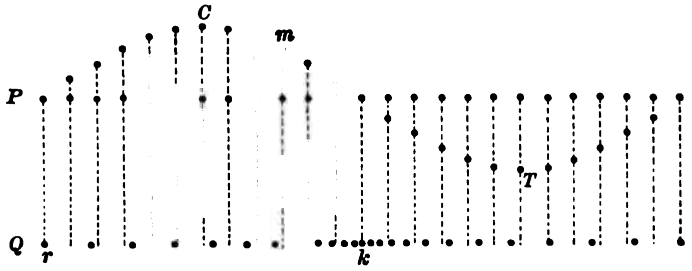


Fig. 334.

Since the forces of restitution requisite for a transverse vibration arise from the shearing elasticity of the medium, transverse waves can be transmitted only by solids.

Light-waves are, however, transverse; but the vibrations in this case must be regarded as of an electrical nature.

467. Longitudinal Waves. — If the displacements of P (Fig. 334) be plotted parallel to the direction of propagation, the particles will now have the arrangement shown at Q, which represents the position of a row of particles, normally equidistant, at any instant while transmitting a longitudinal wave.

At r is a point of rarefaction, or minimum pressure, and at k is a point of condensation, or maximum pressure. An instant later both of these points will have moved to the right, if the differences of phase are regarded the same as in P. The forces of restitution necessary for the propagation of such a wave are those arising from the compressural elasticity of the medium. Accordingly, all forms of matter are capable of transmitting waves of rarefaction and condensation, of which sound-waves are the most important example.

Since a clearer representation of a longitudinal wave may be obtained by plotting either the displacement or the pressure as ordinates, that method will be adopted in the following articles.

468. Equation of a Wave. — To find the equation of a wave, let h, i, j, k, etc. (Fig. 335), be a continuous row of

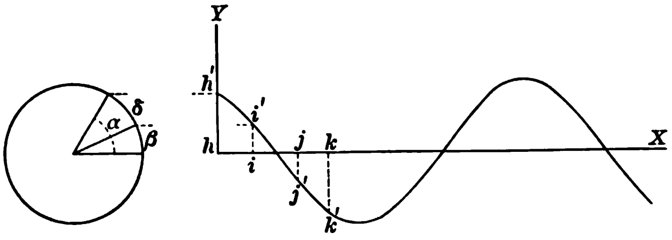


FIG. 335.

particles which at any instant are displaced to the positions h', i', j', k', according to the harmonic law. Take the origin of coördinates at the point h, and suppose that time is reckoned from the instant when h is passing its mean position. The displacement of this point at the time t may then be written

$$hh' = a \sin \frac{2\pi t}{P} = a \sin \alpha,$$

where a is the amplitude, P the period, and α the angle in the circle of reference.

The displacement of the point i at the distance x from the origin will be

$$y = a \sin \beta = a \sin (\alpha - \delta), \tag{3}$$

where β is an angle less than α by an amount, δ, in case the wave is moving to the right.

If the wave-length be denoted by λ,

$$\frac{\delta}{2\pi} = \frac{x}{\lambda}, \tag{4}$$

whence, by substitution in equation 3

$$y = a \sin 2\pi \left(\frac{t}{P} - \frac{x}{\lambda}\right), \tag{5}$$

which is the equation sought. The number δ determines what is called the difference of phase between the points h and i. It is variously referred to as a fraction of the period, as a fraction of the wave-length, or as an angle in the circle of reference.

469. Addition of Waves. — Since, when two displacements of a particle occur simultaneously, neither is modified by the

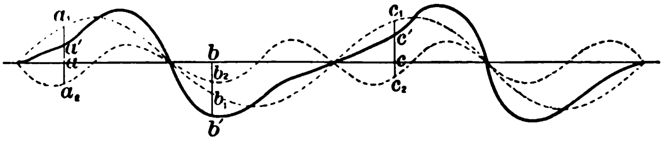


FIG. 336.

other, as was shown in Art. 15, the resultant, arising from the superposition of two systems of waves, may be obtained by the geometric addition of the individual displacements of each particle. For instance, let $a_1b_1c_1$ (Fig. 336) and

$a_2b_2c_2$ be two waves which are simultaneously traversing the row of particles *abc*. Then the displacement *aa'*, *bb'*, *cc'*, of the resultant wave may be found by taking the vector sum $aa_1 + aa_2$; $bb_1 + bb_2$, etc.

Several special cases are of interest.

1°. Two waves of the same length, but differing in phase, combine to produce a wave of the same length, for the dis-

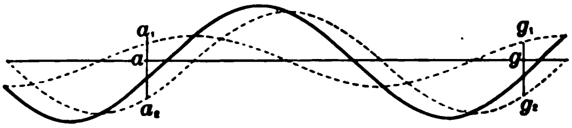


FIG. 337.

placement of any particle, *a* (Fig. 337), in the resultant wave, due to the addition of the displacements aa_1 and aa_2, will be exactly repeated at every other point, *g*, which is distant one or more wave-lengths from *a*.

2°. Two waves of the same period and amplitude, but

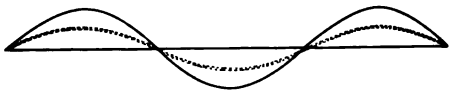


FIG. 338.

differing in phase by a wave-length, combine (Fig. 338) to produce a wave of double amplitude.

3°. Two waves of the same period and amplitude, but

FIG. 339.

differing in phase by a half wave-length, mutually annul each other (Fig. 339). The waves in this case are said to interfere.

4°. Two waves which differ slightly in length combine to produce a wave of varying amplitude as illustrated by the heavy line in Fig. 340.

In the case of sound-waves this system gives rise to the phenomenon of beats.

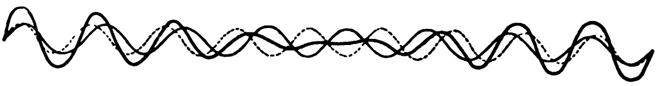


FIG. 340.

The preceding results may also be obtained analytically. Thus, to add two waves having the same period, write $\frac{2\pi}{P} = \omega$ in equation 5, then

$$y_1 = a_1 \sin(\omega t - \delta_1), \tag{6}$$

$$y_2 = a_2 \sin(\omega t - \delta_2). \tag{7}$$

Expanding and adding,

$$y_1 + y_2 = \sin \omega t \cdot (a_1 \cos \delta_1 + a_2 \cos \delta_2) - \cos \omega t \cdot (a_1 \sin \delta_1 + a_2 \sin \delta_2). \tag{8}$$

Introducing two new symbols, defined by

$$\begin{cases} A \cos \gamma = a_1 \cos \delta_1 + a_2 \cos \delta_2 \\ A \sin \gamma = a_1 \sin \delta_1 + a_2 \sin \delta_2, \end{cases} \tag{9}$$

$$\begin{aligned} y_1 + y_2 &= A \sin \omega t \cos \gamma - A \cos \omega t \sin \gamma \\ &= A \sin(\omega t - \gamma); \end{aligned} \tag{10}$$

whence it is seen that two sine waves having a common period combine to produce a new sine wave of the same period, but with amplitude A and phase constant γ, the values of which from equations 9 are

$$A = \sqrt{a_1^2 + a_2^2 + 2a_1 a_2 \cos(\delta_1 - \delta_2)}, \tag{11}$$

$$\tan \gamma = \frac{\Sigma a \sin \delta}{\Sigma a \cos \delta}. \tag{12}$$

When one wave is a whole wave-length behind the other, or the difference of phase is any multiple of 2π,

$$\delta_1 - \delta_2 = 2\pi k$$

(k being some integer), whence

$$A = a_1 + a_2. \tag{13}$$

When one wave is an odd half wave-length behind the other, or the difference of phase is any multiple of π,

$$\delta_1 - \delta_2 = (2k+1)\,\pi, \text{ or}$$

$$A = a_1 - a_2. \tag{14}$$

If, in addition, $a_1 = a_2$, the wave vanishes.

To investigate the case where one period is slightly greater than the other, it will be convenient to write equation 5 in the form

$$y = a \sin 2\pi n\left(t - \frac{x}{v}\right), \tag{15}$$

which may be done by aid of the relations

$$n = \frac{1}{P} \text{ and } v = n\lambda.$$

Suppose, now, that the amplitudes of both waves are the same, but that the frequency of one is m and that of the other is n. The equation of the resultant wave is then

$$y_1 + y_2 = a \sin 2\pi m\left(t - \frac{x}{v}\right) + a \sin 2\pi n\left(t - \frac{x}{v}\right). \tag{16}$$

Making use of the trigonometric relation

$$\sin\alpha + \sin\beta = 2 \sin\left(\frac{\alpha+\beta}{2}\right)\cdot\cos\left(\frac{\alpha-\beta}{2}\right), \tag{17}$$

equation 16 may be written

$$y_1 + y_2 = 2a \cos 2\pi\left(\frac{m-n}{2}\right)\left(t - \frac{x}{v}\right)\cdot \sin 2\pi\left(\frac{m+n}{2}\right)\left(t - \frac{x}{v}\right). \tag{18}$$

This equation may be most readily interpreted by writing it in the form

$$A \sin 2\pi N\left(t - \frac{x}{v}\right), \tag{19}$$

which shows that the resultant wave has a frequency

$$N = \frac{m+n}{2},$$

which is the mean of its components. Its amplitude, however, is variable, both in time and place, as is indicated in Fig. 340. The value of A,

$$A = 2a \cos 2\pi \left(\frac{m-n}{2}\right)\left(t - \frac{x}{v}\right), \tag{20}$$

shows that, as the waves pass any given point, their amplitude varies from 0 to $2a$, $(m-n)$ times per second.

470. Stationary Waves.— When two equal waves traverse a row of particles in opposite directions, the resulting wave remains stationary.

Let A (Fig. 341) be a wave moving to the right, and B a similar wave moving to the left. At the point b, midway between the points of zero displacement x and y on each

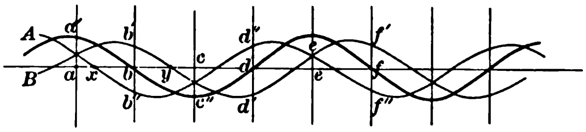


FIG. 341.

wave, draw the ordinates bb' and bb''. Since b' and b'' are homologous points on equal waves, these ordinates are equal, and b will be a point of zero displacement on the resultant wave. Also, since the waves are equal and moving with

the same velocity, the increase of bb' in any given time will be the same as the increase of bb'', and b will remain a point of no displacement; that is to say, the wave remains stationary.

The intermediate points of the row vibrate together from zero displacement to a certain maximum value, depending upon their position in the wave. The points of the row which remain at rest are termed *nodes*, and those which vibrate through the greatest amplitude, distant a quarter wave-length from the nodes, are called *antinodes*.

The analytical expression for a stationary wave is obtained by adding

$$y_1 = a \sin (\omega t - \delta),$$

a wave traveling to the right, to

$$y_2 = a \sin (\omega t + \delta),$$

a similar wave traveling to the left.

Substituting

$$-\delta_2 = \delta_1 = \delta,$$

and

$$a_2 = a_1 = a;$$

in equations 11 and 12

$$A = 2a \cos \delta,$$

and

$$\gamma = 0,$$

which, in equation 10, gives

$$\text{(21)} \quad \begin{cases} y_1 + y_2 = 2a \cos \delta \sin \omega t \\ \qquad\quad\;\; = 2a \cos \dfrac{2\pi x}{\lambda} \sin \omega t, \end{cases}$$

which is the equation of a stationary wave.

It is seen from the absence of the phase constant that each particle is in the same phase; the amplitude, however, varies from point to point, and nodes occur at distances which are multiples of a half wave-length.

471. Waves in a Stretched Cord. — The phenomena of longitudinal and transverse waves may be conveniently studied by the aid of a long helical cord, made by winding hard brass wire about a small rod. If one end be fastened to the wall and the cord be held taut, a smart blow with the finger against the cord will cause a transverse wave to run along

FIG. 342.

the cord (Fig. 342). At the fixed end this wave is reflected and will return to A, but with the important modification that a crest is returned as a trough and *vice versa*. In this case reflection is said to occur with change of phase.

When the cord is attached by means of a long thread, as in Fig. 343, the reflection occurs without change of phase, the wave being returned in the same form as it advanced. A division of the energy, however, occurs at B, depending on the relative masses of the cord and the thread.

By giving to the cord a series of properly timed impulses, the reflected and the advancing waves may be made to combine so as to produce a stationary wave.

The same cord may be made to illustrate longitudinal

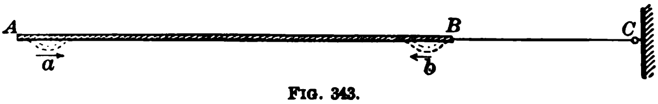


FIG. 343.

waves by plucking the coils apart with a hard point, though in this case the pulse will, in general, move so fast as to be followed with difficulty by the eye. When one end of the cord is fixed, reflection takes place without change of phase; a condensation is reflected as a condensation, and a rarefaction as a rarefaction.

The laws of reflection may be established by consideration of a row of balls (Fig. 344) which may be supposed attached by a series of elastic bonds. If the ball p be struck, a compression will run through the row, and the ball r will fly off; but as it is attached to q by a bond which will transmit a

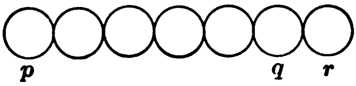


FIG. 344.

tensile stress, it originates a rarefaction which will traverse the row from right to left.

If the row be composed of balls of two sizes, as in Fig. 345, the change in the velocity of the ball b may be found by the equations of impact (Art. 81). Let the masses of the balls

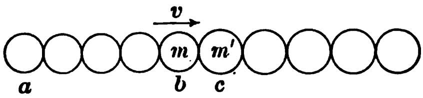


FIG. 345.

be m and m', and their velocities before impact v and 0, and the velocities after impact u and u'. By equations 8 and 9, p. 111,

$$u = \frac{m - m'}{m + m'} v, \text{ and } u' = \frac{2m}{m + m'} v,$$

from which it appears that u is in the same or the opposite direction to v, according as m is greater or less than m'. Accordingly, when a longitudinal wave passes from one medium to another, a part of the energy is reflected and a part transmitted, the phase of the reflected wave being changed, *i.e.* a half wave-length lost, when the reflection occurs in the denser medium. In the rarer medium the phase of the wave is unaltered.

By consideration of a row of balls united by bonds possessing a shearing elasticity (Fig. 346), it may be shown in an analogous manner that the direction of the velocity of a transverse impulse imparted to a row of particles such as m will be reversed or continued at m', according as m is less or greater than m'. The reversal of the velocity of m will evidently return a crest in the reflected wave in place of a trough in the advancing wave.

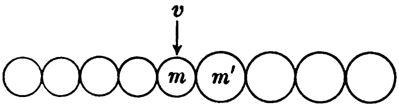


FIG. 346.

Hence, whenever a transverse wave passes from one medium to another, a part of the energy is reflected and a part transmitted, the phase of the reflected wave being changed, *i.e.* a half wave-length lost, when the reflection occurs in the rarer medium. In the denser medium the phase of the wave is unaltered.

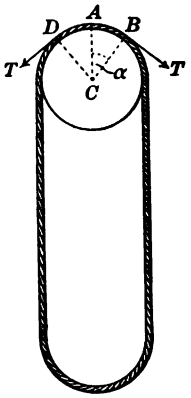


FIG. 347.

472. Velocity of a Transverse Wave. — Let $DABH$ (Fig. 347) be a perfectly flexible massive cord passing about a pulley, C, and driven with a constant velocity.

Let $T =$ tension in the cord,
$v =$ velocity of the cord,
$R =$ radius of curvature at A,
$l =$ length of small arc DB,
$m =$ mass of the length l of the cord,
$\mu =$ mass per unit length.

The normal force at A, due to the tension, supposed to be applied at B and D, is

$$F = 2T \sin \alpha. \tag{22}$$

The force per unit length, when the point B is regarded as very close to A, is

(23) $$\frac{F}{l}=\frac{2T\alpha}{l}=\frac{T}{R}.$$

But by the law of uniform motion in a circle

$$F=\frac{mv^2}{R},$$

or,

(24) $$\frac{F}{l}=\frac{m}{l}\frac{v^2}{R}=\mu\frac{v^2}{R}.$$

If the velocity of the cord be such that the pressure against the pulley just vanishes,

(25) $$\mu\frac{v^2}{R}=\frac{T}{R},$$

or,

(26) $$v=\sqrt{\frac{T}{\mu}}.$$

Under these circumstances the pulley might be removed without altering the form of the cord at BD, and as all motion is relative, it is a matter of indifference whether the cord be regarded as running around the bend, or the bend be regarded as running around the cord with a velocity, v.

Adopting the latter view, equation 26 shows that a transverse wave will travel along a stretched cord with a velocity equal to the square root of the tension divided by the mass per unit length.

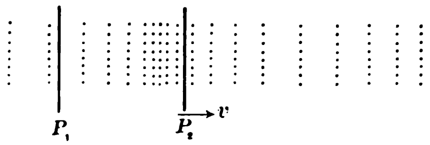


FIG. 348.

473. Velocity of a Compressural Wave. — Let Fig. 348 represent a portion of a medium through which a compressural wave is passing from left to right, and suppose that P_1, P_2 are

two planes which, keeping a constant distance apart, move along with the velocity of the wave so as always to maintain the same relative position with respect to it. Under these circumstances a certain quantity of matter may be regarded as flowing through any area, A, of the planes. To find its amount

Let $u_1 =$ the actual velocity of vibration of a particle close to P_1,
$m_1 =$ the mass which flows through the area A in P_1,
$\beta_1 =$ the volume per unit mass near P_1,
$v =$ velocity of the wave,
$\tau =$ an interval of time so short that u_1 remains sensibly constant.

Then, since the velocity of the matter with respect to P_1 is $u_1 - v$, the length of the portion of the substance which passes in the time τ is $(u_1 - v)\tau$, and its mass is

$$m_1 = \frac{(u_1 - v)\tau \cdot A}{\beta_1}; \tag{27}$$

similarly, the mass which passes P_2 will be

$$m_2 = \frac{(u_2 - v)\tau \cdot A}{\beta_2}. \tag{28}$$

Now, since the distance between the planes is invariable, and the density of the intercepted substance does not change, for the planes always occupy the same position with respect to the wave, as much matter will pass in through P_2 as passes out through P_1. Therefore, $m_1 = m_2 = m$, say.

Making these substitutions,

$$u_1 = v - \frac{m\beta_1}{\tau A}, \tag{29}$$

$$u_2 = v - \frac{m\beta_2}{\tau A}. \tag{30}$$

Now, although the same quantity of matter issues from the plane P_1 as entered at P_2, it suffers a change of momentum from the operation of the force of elasticity of the medium. Calling this force F, and measuring it by the rate at which it changes the momentum,

(31) $$F = \frac{mu_1 - mu_2}{\tau} = \frac{m^2}{A\tau^2}(\beta_2 - \beta_1), \text{ by equation 30.}$$

Again, in the definition of the coefficient of elasticity, $E = \frac{\text{stress}}{\text{strain}}$, if β is the volume of unit mass of the undisturbed medium, the strain will be $\frac{\beta_2 - \beta_1}{\beta}$;
whence

(32) $$E = \frac{F}{A}\frac{\beta}{\beta_2 - \beta_1} = \frac{m^2}{A^2\tau^2}\beta, \text{ by equation 31.}$$

But for the undisturbed medium

(33) $$m = \frac{v\tau A}{\beta},$$

which, substituted in equation 32, gives

$$E = \frac{v^2}{\beta}.$$

Replacing $\frac{1}{\beta}$ by the density ρ, and solving for v,

(34) $$v = \sqrt{\frac{E}{\rho}},$$

which is the velocity of a compressural wave in any medium, provided the temperature remains constant. If the transmission is adiabatic, this formula becomes

(35) $$v = \sqrt{\gamma\frac{E}{\rho}},$$

where γ is the ratio of the two specific heats, as explained in Art. 240.

474. Composition of Vibrations. Lissajous's Figures. — A case of composition of vibrations of considerable interest is that in which a particle is displaced by two harmonic vibrations in directions at right angles, particularly when the periods of the two motions are in the ratio of two small whole numbers.

The curves described by the particle may be found by eliminating t from the equations

$$x = a \sin (2\pi nt - \delta), \tag{36}$$

$$y = b \sin 2\pi n't. \tag{37}$$

The character of the different curves may be more readily studied by the following geometric construction.

Suppose first, for simplicity, that both periods of vibration are the same, and that they take place, respectively, in the lines RQ and TS (Fig. 349).

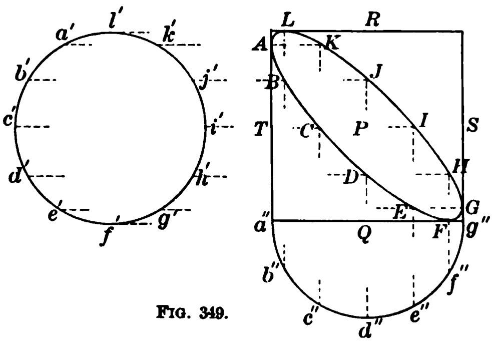


FIG. 349.

Let $a'b' \cdots k'l'$ be the circle of reference for the first vibration, and $a''b'' \cdots k''l''$ that for the second. Suppose, also, that the difference of phase is such that when the vertical displacement of the vibrating point P is TA, the horizontal displacement is TP. The actual position of the point will then be A, which may be determined by the abscissa through a' and the ordinate through a''. In like manner b' and b'' determine B, the position of P after a short lapse of time, with similar constructions for other points. The resulting curve for this case

is an ellipse inscribed in a rectangle having sides equal to twice the respective amplitudes.

The effect of altering the difference of phase will be to shift the position and dimensions of the ellipse, leaving it always inscribed in the same rectangle. When the ratio of the periods is 2 to 1, an application of the preceding method gives the curve of Fig. 350.

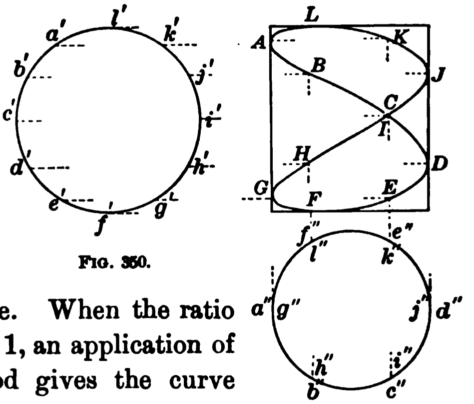


FIG. 350.

Fig. 351 shows some of the forms assumed as the difference of phase is changed from 0 to π when the ratios

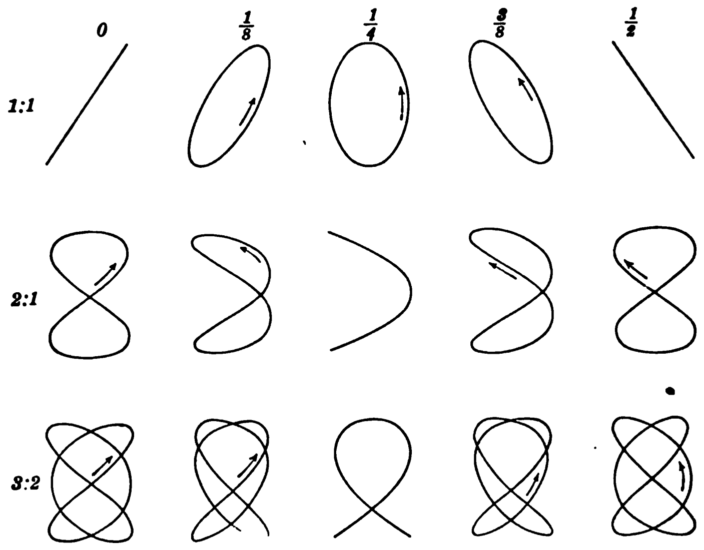


FIG. 351.

are $\frac{1}{1}$, $\frac{2}{1}$, or $\frac{3}{2}$. From π to 2π the curve runs through the same series in the inverse order.

As the values of n and n' are increased, the figures become very complicated; there is, however, a method of regarding them, introduced by Lissajous, after whom they are named, which affords a clear conception of the changes the figures undergo as the periods are altered.

Let one of the directions of vibration of the point P, for the case where the periods are equal, be RQ (Fig. 352), and suppose that the other is replaced by uniform motion in a circle $TCSI$. The combination of these two motions will give a curved path, $LCFI$, lying upon the surface of a cylinder.

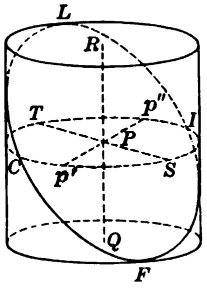


FIG. 352.

If the eye be placed at a great distance in the plane of TCS, the horizontal motion of the point P will appear rectilinear, and the curve $LCFI$ as one of the forms of the ellipse (Fig. 349). By revolving the cylinder about its axis, the projected curve may be made to pass through all the forms which would be obtained by varying the difference of phase of two rectilinear vibrations between 0 and 2π.

If the surface of the cylinder be unrolled, it is obvious, from the mode of generation of the curve $LCFI$, that the latter will be developed as a simple sine curve.

From the preceding principles it appears that Lissajous's figures may be regarded as the projection of sine curves traced upon the surface of a cylinder.

475. Water Waves.—When the surface of a body of water, or other liquid, is disturbed, a system of waves is formed of a type essentially distinct from those previously described.

Although for small amplitudes water waves are hardly distinguishable from those arising from transverse vibrations, it is obvious that they cannot be of this type, since the lower layers of the liquid must then be alternately rarefied and compressed. The actual motion of the particles of water in transmitting a surface-wave of considerable dimensions, say longer than a foot, is one of revolution in a vertical curve under the influence of weight, as appears on observation of a moving liquid carrying fine particles of any substance in suspension. In deep water these curves are exactly circles.

476. Lyman's Apparatus. — Many interesting properties of water waves are exhibited in the apparatus shown in Fig. 353, designed by Professor C. S. Lyman.

The particles of water at the surface are represented by little studs, A, B, · · · , attached to a series of cranks, which

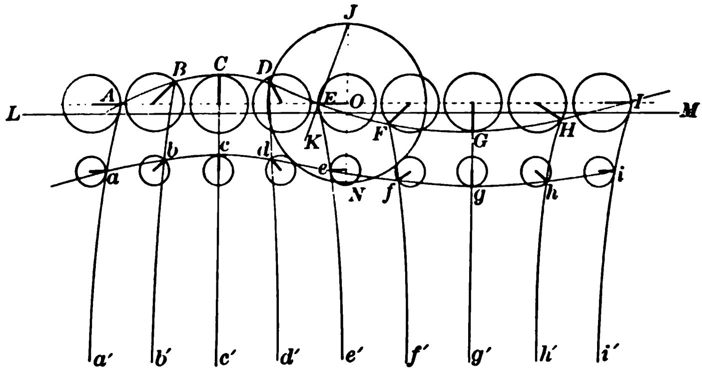


FIG. 353.

revolve simultaneously, and the wave itself by a flexible wire passing through holes drilled in these studs.

The motion of the particles below the surface diminishes according to the exponential law. At a depth of an eighth

of a wave-length the excursions of the particles represented by a, b · · · are only half as great as at the surface, and at the depth of a half wave-length the water is virtually quiescent.

The wave form belongs to a family of curves known as trochoids, and may be generated by rolling a circle on the

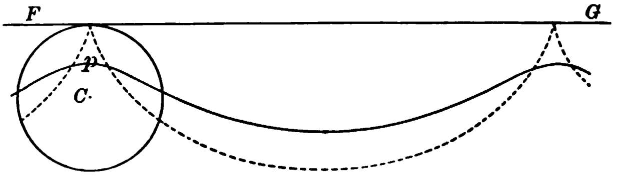


FIG. 354.

line FG (Fig. 354), while some point, p, is allowed to trace its own path. The curve is never symmetrical with respect to the line of centers, the maximum curvature always being numerically greater than the minimum.

When the describing point is chosen on the circumference of the rolling circle, the roulette takes the special form known as the cycloid, the part in this case corresponding to the crest of the wave rising to a cusp.

JN (Fig. 353) is a circle, having a circumference equal to the length of the wave. It may be shown that the period of the wave is that of a pendulum whose length is equal to the radius of this circle. Calling this radius R, the wave-length λ, and the period P,

(38) $$R = \frac{\lambda}{2\pi},$$

(39) $$P = 2\pi\sqrt{\frac{\lambda}{2\pi g}} = \sqrt{\frac{2\pi\lambda}{g}},$$

whence

(40) $$v = \frac{\lambda}{P} = \sqrt{\frac{g\lambda}{2\pi}},$$

or the velocity of the wave increases as the square root of the wave-length.

The wire JK in the figure, pivoted at J and constantly passing through the point E, has always the direction of the resultant force passing through the particle E; for, letting r denote the radius OE, the central force, acting on the particle m at E, will be

$$F_c = 4\pi^2 \frac{mr}{P^2},$$

which, combined with the expression for P in equation 39, gives

$$\frac{mg}{F_c} = \frac{R}{r}; \tag{41}$$

whence it follows that, if EO is taken proportional to the central force on the particle, and JO to the weight, JE will represent in magnitude and direction the resultant force on the particle.

The model also shows that the direction of this force is always at right angles to the surface of the wave, as it should be.

When $F_c = mg$, the particles at the pointed crest of a wave lose their weight and are readily blown off in the wind.

Since in forming a wave more water is taken out of the trough than is put into the crest, the line of centers is somewhat higher than the original surface LM.

The vertical wires Aaa', $Bbb' \cdots$ of the model illustrate the peculiar swaying motion which takes place in lines initially vertical, or the strain to which a vertically floating board would be subjected.

Similarly, the bending of the wire $ABCDEFG$ shows the strain on a horizontal body at the surface.

The distortion suffered by a rectangular block of water is shown by $ABba$, $BCcb$, etc., which assists one to understand why a wrecked vessel is so quickly broken up by the waves.

477. Ripples.—In the previous discussion of water waves it has been assumed that the waves were of considerable size. In the case where their dimensions are not of such size that the weight of water displaced in each wave may be regarded as very great with respect to the force of surface tension, the expression for the wave velocity must be altered by the addition of a term containing this force. The equation then takes the form

$$v=\sqrt{\frac{g\lambda}{2\pi}+\frac{2\pi T}{\rho\lambda}}, \tag{42}$$

where T denotes the surface tension and ρ the density of the liquid. It thus appears that the velocity of the wave will increase with diminishing wave-length, if this is very small, and also with increasing wave-length, if this is large. When the two terms are equal, *i.e.* when

$$\lambda_o^2=\frac{4\pi^2T}{\rho g}, \text{ or } v_o^2=2\sqrt{\frac{gT}{\rho}},$$

the wave will have a minimum velocity.

This minimum velocity for water is about 23 $\frac{\text{cm.}}{\text{sec.}}$, which corresponds to a wave-length of 1.72 cm.

Wavelets in which the surface tension is the preponderating force, that is, those whose wave-lengths are less than 1.7 cm., are designated as *ripples*.

The existence of a minimum velocity for water waves offers a simple explanation for the singular fact that the surface of a small body of water may remain perfectly calm, although the air above it is not so; for, if a variation in the air pressure should create a slight wave disturbance, the ripples would run away from the disturbing cause as long as the velocity of the air currents remained less than 1.7 $\frac{\text{cm.}}{\text{sec.}}$. It appears further, from equation 42, that there are in gen-

eral both a ripple and a wave, which travel with the same velocity.

These may be shown experimentally by moving a small rod dipped vertically into water, with a velocity slightly greater than 23 cm. per sec. The ripples in this case may be observed advancing in a group with, and in front of, the rod, while the waves follow at the same speed behind. If the motion of the rod be quickened, the wave-length of the ripples will be shortened, while that of the waves is increased.

The dependence of the wave-length and velocity of ripples upon surface tension has been utilized by Lord Rayleigh to determine the value of surface tension in various liquids. The method in brief is to produce a series of stationary ripples by the vibrations of a style, attached to a tuning fork of known period, and dipping into a trough of the liquid. The wave-lengths are then carefully observed and the surface tension calculated from these data.

478. Wave Propagation in Three Dimensions. — Suppose a disturbance takes place at a point, p (Fig. 355), in an isotropic medium. Since it will be propagated with the same velocity in all directions, after a certain lapse of time the disturbance will have spread so as to affect similarly a series of particles lying in the surface of a sphere, *abc*. A disturbance propagated in this manner is termed a spherical wave, the wave front being defined as the locus of all particles having the same phase.

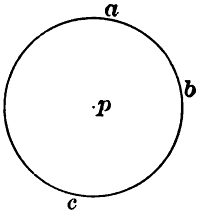


FIG. 355.

479. Huyghens's Principle. — Suppose that, when the wave originating at p has reached the point b (Fig. 356), the vibrations of every other particle are reduced to rest. The point

b now becomes a center of disturbance, and an instant later a small wave from this center will have reached b'. But the motion of b is just the same whether the neighboring particles are at rest or vibrating. Therefore, every point of the wave front passing through $bcde$ must be regarded as a source of disturbance, and as sending out little waves which an instant later will have reached $b'c'd'e'$. But, since the only wave which can be physically recognized at $a' \cdots b' \cdots g'$ is the spherical one having the center p, it follows that any wave may be regarded as emanating either from an original source, p, or from a series of points through which the wave front passed an instant before.

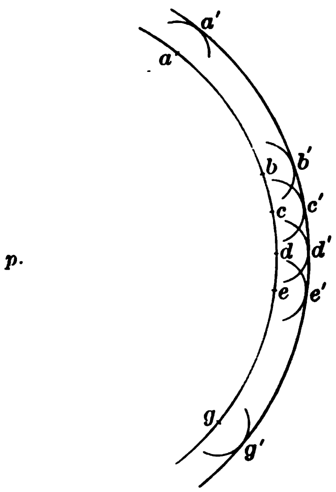


FIG. 356.

This mode of regarding the disturbance at any point of a wave front as the resultant of the separate disturbances which would have been produced by each point in the wave front acting singly at some earlier position, is known as Huyghens's Principle.

Stokes has shown by analysis that each of the elementary waves mutually destroy each other, except at the surface which is their envelope.

480. Reflection and Refraction. — Suppose that AB (Fig. 357) represents the smooth interface between two media of different density, the denser one being below. By a smooth surface is meant one in which the roughnesses are small

compared to the wave-lengths. Thus, a masonry wall would be smooth in this sense to long sea waves, or ordinary sound waves, but to such short waves as produce the sensation of vision, only the most highly polished surfaces can properly be called smooth.

Suppose, also, that kl and op are the fronts of a train of waves moving in the direction of the arrow. Experience shows that when op strikes AB there arise, in general, two new systems of waves with changed directions; those such as oq, moving in the first medium, are known as *reflected* waves, and those such as oq', transmitted through the medium on the other side of the bounding surface, are called *refracted* waves.

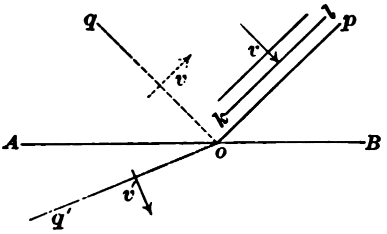


FIG. 357.

To determine the laws of reflection and refraction it will be sufficient to observe that at the surface AB there is a portion of the wave surface which is common to the three systems. Let the angle between the incident wave po and the interface, namely, poB, be denoted by i; the angle between the reflected wave oq and AB, namely, qoB, be denoted by ρ; and the angle between the refracted wave oq' and AB, namely, Aoq', be denoted by r. Also, call the velocity of the waves in the first medium v, and in the second v'; then, since the velocity in the direction of the interface of that portion of the wave common to all is the same in each system,

$$\text{(43)} \qquad \frac{v}{\sin i} = \frac{v}{\sin \rho} = \frac{v'}{\sin r},$$

whence

$$\text{(44)} \qquad \sin \rho = \sin i,$$

and

$$\frac{\sin i}{\sin r} = \frac{v}{v'}. \tag{45}$$

From equation 44, since i is not identically equal to ρ, for there is known to be a change of direction,

$$\rho = \pi - i, \tag{46}$$

which is the general law for reflection for all types of wave surfaces.

It is frequently stated by saying that the angle of incidence is equal to the angle of reflection; but this statement, in strictness, requires the additional explanation that the angle of reflection is to be measured in the opposite direction from the angle of incidence.

Equation 45 is the law of refraction. Since the ratio of the velocities in the two media is constant, it may be denoted by a single letter, say n, which is called the *index of refraction.* Writing the law in the ordinary form,

$$\sin i = n \sin r. \tag{47}$$

Instances of reflection and refraction are common in the phenomena of sound, light, radiant heat, so called, and in electrical waves; but, since the direction of propagation of light-waves only can be accurately determined by our senses, the general discussion of the consequences of these laws may be postponed to the chapters on light.

CHAPTER XXXI.

SOUNDS AND THEIR RELATIONS.

481. Sound. — Sound may be defined as a sensation peculiar to the auditory nerves.

A sound sensation is normally produced by a series of waves originated by a vibrating body and transmitted to the ear by an elastic medium, usually the air.

A sounding body is easily recognized to be in vibration. Thus, for example, a stretched string when sounding takes the appearance of an elongated spindle with an indistinct or hazy appearance about the middle. If a light body, like a pith-ball, be arranged so as to rest against a sounding rod or plate, the ball will be visibly agitated. The sound in each case ceases if the body is touched in such a way as to stop its vibration. The sounding column of air in an organ pipe may be shown to be in vibration by lowering a horizontal membrane, covered with a few grains of sand, by a thread into a pipe furnished with a glass window. When the pipe is made to speak, the sand is, in general, thrown into violent agitation.

To show that air is the ordinary medium of transmission of sound-waves, it is sufficient to place a bell, actuated by clock work, under the receiver of an air pump, taking care to support it so that its vibrations cannot be transmitted through the attachment. On exhausting the receiver, the sound grows fainter and fainter, and at last practically ceases. If the receiver be first filled with hydrogen, and then exhausted, the diminution of sound will be more marked, the

reason being that less energy is transmitted to the glass by hydrogen than by air, on account of the difference in density (Art. 471).

The ear broadly distinguishes *noises* and *musical sounds.* The former arise from vibrations which are entirely irregular; the latter result from periodic vibrations.

It is obvious that sound-waves in air must be of the compressural type, since no other can be transmitted by a fluid.

482. Characteristics of a Musical Sound.—The ear recognizes a musical sound to be capable of variation in three particulars, known as *pitch*, *loudness*, and *quality*.

483. Pitch.—The pitch of a note is determined by the frequency or number of vibrations per second, an acute note having a greater frequency than a grave one. Since velocity, wave-length, and frequency are connected by the equation $v = n\lambda$, any given note may also be designated by its wave-length in air.

484. Loudness.—The loudness or intensity of a sound is found to depend upon the energy of the vibrations transmitted to the ear, and to increase with it. As a first approximation, the loudness of a note may be taken as proportional to the amplitude of its vibration.

It is obviously impossible to state a sensation with great precision, but experiment indicates that a more exact expression for the loudness of a sound involves the pitch of the note, and the logarithm of the energy reaching the ear.

485. Quality.—Difference in quality or *timbre* is a term used to express the distinction between notes of the same pitch and equal loudness, emitted by different instruments, such as, *e.g.*, the violin, the clarinet, or the human voice.

The quality of a note was first shown by Helmholtz to be determined by the presence of a series of notes simply related to the fundamental or tone of lowest pitch.

Plotting the pressure at any point, in a medium transmitting sound-waves, as a function of the time, the pitch of a given note may be represented by the length, the loudness by the amplitude, and the quality by the form of the wave.

486. Siren.—A convenient instrument for producing notes of any required pitch is the siren, invented by Caignard de la Tour. It consists of a wind chest, *A* (Fig. 358), having a top perforated with a circular row of holes. Close to this, on the upper side, is a metallic disc, *B*, pierced in an analogous manner, and so arranged that it will rotate freely on a vertical axis. The direction of the holes is inclined to the face of plates, those in the disc sloping in a direction opposite to those in the top. Air, on being forced into the chest by a bellows, escapes through the openings in the top; but since, in its passage, the direction of the current is changed, a certain pressure will be exerted against the sides of the holes, setting the disc in rotation. As a result of this rotation, the air escapes in a succession of puffs which give rise to musical sounds of definite pitch. In order to count the number of revolutions, the spindle *C* carries at the upper portion a screw which can be made to engage with a wheel of one hundred teeth, *D*, to which is attached a pointer moving over a graduated dial. By the side of *D*, and connected with it, so that it is advanced one tooth for every revolution of *D*,

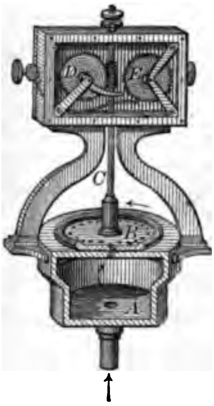


Fig. 358.

is a similar wheel, E, the whole serving as an automatic counter of the number of revolutions made by the perforated disc. In order to study notes which are simply related, the siren is usually made with more than one row of holes in the disc, and provided with stops, so that the air can be admitted to one or all rows at pleasure. When it is desired to obtain the frequency of a note, wind is admitted to the chest until the note emitted by the siren is judged to be in unison with the given note, after which the speed is kept as nearly constant as possible, and the train of counters thrown in gear with the spindle for a certain period of time, which is carefully noted. The product of the number of holes in the row used, by the number of revolutions per second, gives the frequency of the note sounded.

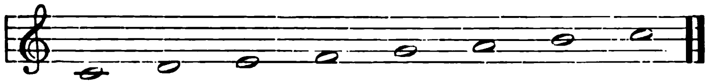
As it is quite possible to make an error of one revolution, such a determination may be in doubt as much as ten vibrations.

487. Vibroscope. — A tuning fork may be made to record its frequency automatically by attaching a style to one of the prongs so that it will leave an undulating trace upon a piece of smoked paper wrapped about a revolving cylinder, as in Fig. 3. By arranging a pendulum so as to close an electric circuit every second, a spark is made to pierce the paper at the point of the style and mark equal intervals of time. The frequency of the fork is thus readily found by counting the number of undulations between the punctures. The free period of the fork will be influenced slightly by the mass of the style, its friction against the paper, and, when the fork is electrically driven, by the forced character of the vibrations. If, however, the period of a particular fork be carefully determined under definite conditions from its graphical record, the frequency of any other fork may be obtained from it with great accuracy by Lissajous's method of comparison (Art. 510).

488. Musical Intervals. — If the outer and inner rows in the siren, having, respectively, sixteen and eight holes, be open at the same time, the resulting tones are recognized to be in a concordant relation to each other, called the octave, whatever the absolute pitch of the notes may be. In general, when two notes differ in pitch, they are said to be separated by a *musical interval*, which is measured by the ratio of the frequency of the higher to that of the lower note. If in the siren the rows containing twelve and eight holes are opened, the ear will recognize an interval known as the Fifth. Similarly, the rows of sixteen and twelve holes, respectively, will yield the Fourth.

489. Musical Scales. — It has been recognized as a fundamental principle of the musical compositions of all nations that the alteration of pitch in any melody takes place by steps, or intervals, and not by continuous transition. The particular succession of intervals by which a composition advances from one note to its octave is called a *musical scale*.

The major scale is a succession of eight notes which are related to the fundamental and variously designated as shown in the following scheme, the first note being taken as 256



256	288	320	341	384	427	480	512
c′	*d′*	*e′*	*f′*	*g′*	*a′*	*b′*	*c″*
Do	Re	Mi	Fa	Sol	La	Si	Do
$\frac{1}{1}$	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	$\frac{2}{1}$
First.	**Second.**	**Third.**	**Fourth.**	**Fifth.**	**Sixth.**	**Seventh.**	**Octave.**

vibrations per second, which is practically the middle C of the piano, or the fundamental tone of an organ pipe two feet long.

The first three methods of representation determine the absolute pitch of the notes, but the last three refer only to their relative positions on the scale. The intervals between each note and the next higher are:

Major Second	Minor Second	Half-Tone	Major Second	Minor Second	Major Second	Half-Tone
$\frac{9}{8}$	$\frac{10}{9}$	$\frac{16}{15}$	$\frac{9}{8}$	$\frac{10}{9}$	$\frac{9}{8}$	$\frac{16}{15}$

The smallest interval recognized in music is $\frac{81}{80}$, the difference between the major and minor seconds. It is called a *comma.*

The various other octaves are usually designated by subscripts and accents as follows:

C_1 (16 ft.)	C (8 ft.)	c (4 ft.)	c' (2 ft.)	c'' (1 ft.)	c''' ($\frac{1}{2}$ ft.)	c^{IV} ($\frac{1}{4}$ ft.)	c^{V}
32	64	128	256	512	1024	4096	8192

The notes of the scale are often changed by an interval of a *chromatic semitone.* Theoretically this interval may have two values: $\frac{25}{24}$, the minor semitone; and $\frac{25}{24} \cdot \frac{81}{80} = \frac{135}{128}$, the major semitone ; but the difference is hardly perceptible by most ears, and the ordinary musical notation does not distinguish between them. A note changed by a semitone is called *sharp* when raised, and *flat* when lowered, *e.g.* B when raised a semitone is designated by B♯, read B sharp, and when lowered by B♭, read B flat. The scale of twenty tones, obtained by introducing these new tones, is called the *chromatic scale.*

In addition to the two scales already defined, there is a group of three scales designated collectively as the *minor mode.* They are, in the key of C:

C		D		Eb		F		G		Ab		B		C
1		$\frac{9}{8}$		$\frac{6}{5}$		$\frac{4}{3}$		$\frac{3}{2}$		$\frac{8}{5}$		$\frac{15}{8}$		$\frac{2}{1}$
	$\frac{9}{8}$		$\frac{16}{15}$		$\frac{10}{9}$		$\frac{9}{8}$		$\frac{16}{15}$		$\frac{75}{64}$		$\frac{16}{15}$	

C		D		Eb		F		G		A		B		C
1		$\frac{9}{8}$		$\frac{6}{5}$		$\frac{4}{3}$		$\frac{3}{2}$		$\frac{5}{3}$		$\frac{15}{8}$		$\frac{2}{1}$
	$\frac{9}{8}$		$\frac{16}{15}$		$\frac{10}{9}$		$\frac{9}{8}$		$\frac{10}{9}$		$\frac{9}{8}$		$\frac{16}{15}$	

C		D		Eb		F		G		Ab		Bb		C
1		$\frac{9}{8}$		$\frac{6}{5}$		$\frac{4}{3}$		$\frac{3}{2}$		$\frac{8}{5}$		$\frac{9}{5}$		$\frac{2}{1}$
	$\frac{9}{8}$		$\frac{16}{15}$		$\frac{10}{9}$		$\frac{9}{8}$		$\frac{16}{15}$		$\frac{9}{8}$		$\frac{10}{9}$	

The first is usually an ascending instrumental scale, the second an ascending vocal scale, and the third a descending scale, though usage is not perfectly settled in these respects.

The one interval common to all, and that which gives a minor chord its peculiar character, is the minor third, $\frac{6}{5}$.

490. Transposition. — In order to accommodate different voices or instruments, the Do or keynote of a composition may be transferred from its first position to any other note of the scale. Suppose, for instance, it is desired to begin a scale on $D = \frac{9}{8}n$. Then, calculating the frequencies and comparing them with the key of C, it appears that besides the keynote D, the G and B are right, and that E and A differ from the true scale only by the negligible interval $\frac{81}{80}$. The notes

	Key of D.		Changes in Key of C.
Do	$\frac{9}{8}$	=	$\frac{9}{8}$
Re	$\frac{81}{64}$	=	$\frac{81}{80}\cdot\frac{5}{4}$
Mi	$\frac{45}{32}$	=	$\frac{81}{80}\cdot\frac{25}{24}\cdot\frac{4}{3}$
Fa	$\frac{3}{2}$	=	$\frac{3}{2}$
Sol	$\frac{27}{16}$	=	$\frac{81}{80}\cdot\frac{5}{3}$
La	$\frac{15}{8}$	=	$\frac{15}{8}$
Si	$\frac{135}{64}$	=	$\frac{81}{80}\cdot\frac{25}{24}\cdot 2$
Do	$\frac{9}{4}$	=	$\frac{9}{4}$

F and C, however, must be raised a chromatic semitone before they can be introduced into the new scale. Instead of writing the sharps before these notes every time they occur, they are usually placed at the beginning of the staff, forming what is called the signature of the key, thus:

491. The Tempered Scale. — In order to perform a piece of music in just intonation in all the keys in present use, a scale containing at least fifty-three tones would be required. To voices, strings, or trombones this is of no moment, but in any instrument of fixed tone, such as the piano or organ, it is practically impossible to control so many notes. For these instruments a compromise scale, known as the *scale of equal temperament*, is used. In it the octave is divided into twelve equal intervals represented by the number $\sqrt[12]{2} = 1.059$, thus ignoring the difference between the major and minor tones, and identifying the neighboring chromatics. The following table exhibits the difference between the true and the tempered scale.

	Do	Re	Me	Fa	Sol	La	Si	Do
Just	256	288	320	341.3	384	426.7	480	512
Even	256	287.3	322.5	341.7	383.6	430.5	483.2	512

The only accurately tuned interval in such a scale is the octave, the others being more or less false. In a melody the difference is hardly noticeable, but when several notes are sounded in a chord, especially on an instrument giving sustained tones, the contrast between the tempered and the true scale is more marked. Music rendered in the scale of even temperament is decidedly inferior to that played in just intonation.

492. Limits of Audible Sound. — The greatest and the least numbers of vibrations which are capable of producing a musical sound vary somewhat with the individual ear and its age. According to Helmholtz the gravest note, having a definite pitch, is about thirty vibrations per second. Below this, the auditory nerves are no longer excited uniformly throughout the whole time of a vibration, though it may be mentioned that some other investigators have thought that they were able to perceive musical sounds as low as sixteen vibrations per second. The range of audible sounds is about ten octaves, say from $C_1 = 32$ to $c^{\text{VIII}} = 32768$, though many persons of fair hearing cannot perceive notes above $c^{\text{VII}} = 16384$.

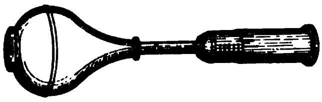
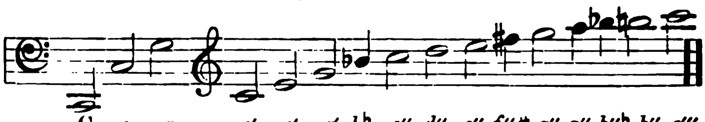


FIG. 359.

By means of a whistle of adjustable length, such as is shown in Fig. 359, it is easy to produce a series of short waves, gradually passing beyond the limit of hearing. The highest note employed in the orchestra is d^{V} on the piccolo.

493. Harmonics. — When the frequency of one note is an exact multiple of another, the second is said to be a harmonic of the first. Thus, the first sixteen harmonics of $C = 64$ are, in the ordinary musical notation,



C	c	g	c′	e′	g′	b♭	c″	d″	e″	f″♯	g″	a″	b″♭	b″	c‴
64	128	192	256	320	384	448	512	576	640	704	768	832	896	960	1024

The sixth, tenth, twelfth, and thirteenth overtones, or upper partials as they are sometimes called, are not used in music, and their positions on the staff can be only approximately marked.

EXAMPLES.

1. Show that the interval from g to d' is a true Fifth.

2. Write the scale beginning Do $= \frac{4}{3}n$.

3. A siren of 15 holes is speeded up till it emits the note g''. How many revolutions per minute does it make? *Ans.* 3072.

4. The frequency of three notes are 504, 630, and 945 per second. What are the intervals between them?

Ans. $\begin{cases} \text{Fourth.} \\ \text{Fifth.} \\ \text{Seventh.} \end{cases}$

CHAPTER XXXII.

PROPAGATION OF SOUND-WAVES.

494. Velocity of Sound-Waves. — The velocity of a compressural wave has been shown in general to be

$$v = \sqrt{\frac{E}{\rho}},$$

where E is the elasticity of the medium under constant temperature, and ρ its density. In the case of sound-waves in air, the changes in volume must be regarded as adiabatic, for which case the formula becomes (Art. 241)

$$v = \sqrt{\gamma \frac{p}{\rho}},$$

p being the pressure and γ the ratio of the specific heats.

For air, at 0° C. and one atmosphere pressure,

$$p_0 = 1.014\ (10)^6 \frac{\text{dynes}}{\text{cm.}^2},$$

$$\rho_0 = 1.293\ (10)^{-3} \frac{\text{gm.}}{\text{cm.}^3},$$

$$\gamma = 1.405\ ;$$

whence

$$v = 331.8 \frac{\text{met.}}{\text{sec.}} = 1089 \frac{\text{ft.}}{\text{sec.}}.$$

The mean of ten careful determinations of the velocity of sound by as many different observers has shown it to be, at zero and a pressure of one atmosphere,

$$v = 331.6 \frac{\text{met.}}{\text{sec.}}.$$

The greatest and least values were 332.4 and 330.6.

Since at the temperature $t°$ C.

$$\frac{p}{\rho}=\frac{p_0}{\rho_0}\left(1+\frac{t}{273}\right)$$

(equation 22, p. 181), the velocity of sound at any temperature in a gas will be

$$(1) \qquad v=\sqrt{\gamma\frac{p_0}{\rho_0}}\sqrt{1+\frac{t}{273}}.$$

This velocity is not sensibly affected by the pitch or the intensity of the sound.

From equation 1 it appears that the velocity of sound in different gases should vary inversely as the square root of the density, a conclusion which is entirely verified by experiment.

495. Doppler's Principle. — If an observer be approaching or receding from a fixed source of sound, it is obvious that the number of waves meeting the ear will be greater or less than the actual frequency of the sound by the number of wave-lengths passed over in a second. Thus, if the actual frequency be n, the apparent frequency n', the velocity of sound v, and the velocity of the observer relative to the source u, it is evident that

$$(2) \qquad n'=n\pm\frac{u}{\lambda}, \text{ or } n\left(\frac{v\pm u}{v}\right),$$

on substituting the value of λ.

The alteration in pitch of a bell or whistle, due to relative motion, may often be observed on a passing locomotive.

496. Interference of Sound. — An illustration of the general principle that two waves in opposite phase, when superposed, annul each other, is afforded by the tuning fork.

If a fork, after being set in vibration, be slowly revolved about its axis close to the ear, or before a resonant cavity, it will be found that there are four positions in which the sound attains a maximum intensity.

The general character of the phenomenon may be understood from Fig. 360. Let A, B be two prongs of a tuning fork, each of which is vibrating in a similar manner, but in the opposite phase, as indicated by the arrows.

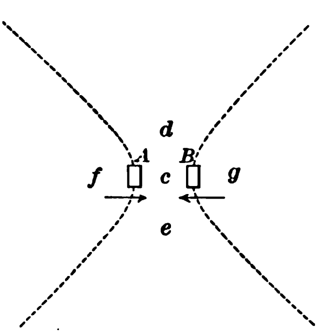


FIG. 360.

Without attempting to state the exact condition of every point in the field, it may be seen that at any point between the prongs their displacements are such as to increase the pressure. Similar conditions obviously obtain for the regions d and e, on each side of the axis AB. In the vicinity of the point f the displacement of A will produce a diminution of pressure, and that of B an increase; but the effect of A will predominate, since this prong is nearer. As one passes from the region f to the region d, there must be a line of equilibrium where there is no change of pressure, and hence extinction of sound. The positions of no sound lie nearly in planes passing through the axis of the fork, and making angles of 45° with its face.

If, while the fork is held so that the ear is on a line of silence, a tube be slipped over one of the prongs without touching it, the sound will be at once restored.

497. Beats. — A beat is a periodic variation in the intensity of a sound produced by the interference of two waves having slightly different periods.

If the pitch of one of two forks which are exactly in unison be lowered by attaching a small coin to one of its prongs, by means of wax, and both be simultaneously sounded, the observer will hear bursts of sound at intervals which decrease as the difference in pitch between the forks is increased. A similar result may be obtained by slightly warming one of the forks.

The mutual reinforcement and destruction of two waves are shown diagrammatically in Fig. 340, Art. 469, where it has also been proved that the number of beats a second is equal to the difference in frequency of the notes, provided they are near together.

Beats may be produced by swinging a fork of acute pitch back and forth in front of a wall. In this case those waves which reach the observer directly, when the fork is moving toward the wall, will, according to Doppler's principle, be slightly lowered in pitch, while, on the other hand, those waves which reach the ear only after reflection from the wall will be slightly raised in pitch. The direct and reflected systems are then in a condition to interfere and produce beats.

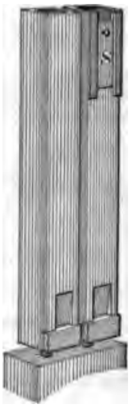


Fig. 361.

If two organ pipes, originally in unison, be brought slightly out of tune by means of a slider, *S* (Fig. 361), which opens a hole at the top of one of the pipes, very strong beats may be produced. When the pipes are exactly in tune, and attached to the same wind chest, the note obtained from the two sounded together is relatively feeble, for the reason that the air columns naturally take a form of alternate vibration, such that if there is a condensation at the mouth of one, there will be a rarefaction at the mouth of the other. The resulting waves differ

by a half wave-length, and hence annul each other. A feather placed at the mouth of either will show that the air makes essentially the same vibrations as if one pipe were sounded alone. If a screen be interposed between the mouths of the pipes, the sound will be restored.

The existence of beats between two notes is a most valuable assistance in the accurate tuning of musical instruments, as the ear possesses but a limited ability to estimate such intervals as the Fifth or Fourth without their aid.

EXAMPLES.

1. What is the wave-length of d^{IV} in air at 0°?
Ans. 2 ft. 1.3 in.

2. At what temperature would the velocity of sound-waves in air be 349 meters per sec.? *Ans.* 28.7°.

3. How long will it take the sound of a signal gun to reach an observer 3.2 miles away if the temperature of the air is 18° C.?
Ans. 15 sec.

4. Calculate the velocity of sound in hydrogen at zero.
Ans. $1.261(10)^5$ cm. / sec.

5. With what velocity would sound travel in water at 11° for which the coefficient of elasticity is $2.1(10)^{10}$ gm. / cm. sec.^{2}?
Ans. $1.45(10)^5$ cm. / sec.

6. Calculate the velocity of sound in steel.
Ans. $0.522(10)^6$ cm. / sec.

7. If the velocity of a sound in a gas be 340 meters per sec. at 16°, what will the velocity be at a temperature of 168° if the pressure of the gas be doubled? *Ans.* 420 met. per sec.

8. If a note sounded on a train, which is approaching an observer at a velocity of 48 miles an hour, has an apparent frequency of 384 vibrations per second, what is its actual frequency?
Ans. 408 per sec.

9. At what velocity must a source of sound approach the hearer in order to raise the pitch a major semitone?
Ans. 40.6 miles per hour.

CHAPTER XXXIII.

SONOROUS BODIES.

498. Vibrating Columns of Air. — The laws of vibrating air columns may be easily deduced from a knowledge of the velocity of a compressural wave and the nature of the reflection from a free or a closed end. Thus, suppose a compression starts at A (Fig. 362) and travels along the tube to B. Supposing the end open, it will be reflected as a rarefaction (Art. 471), which on its return to A is reflected as a condensation and is in the same state as at the start. Since in the time of one vibration it has traveled twice the length L of the tube, the fundamental wave-length is $2L$, and its frequency $n = \frac{v}{2L}$. Whenever a succession of pulses is transmitted along the tube, those advancing combine with those returning to form a stationary wave.

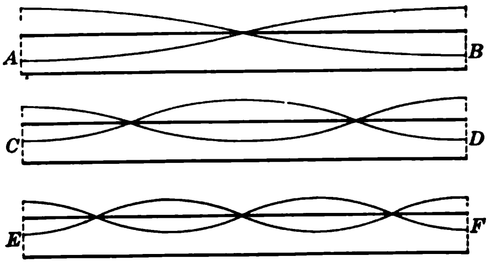


FIG. 362.

The node, or place of maximum variation of pressure, will be at the center, and the antinodes, or places of unchanged pressure, will be at the open ends.

For simplicity of statement it is convenient to consider the displacement of the particles of air at different points along the tube. Drawing these as ordinates above or below

the side of the tube, as in Fig. 362, the position of the node and the length of the wave are clearly shown.

Open tubes may also vibrate with two, three, four, or more nodes, which must be placed so that an antinode will come at the end of the tube. These cases are represented at *CD* and *EF*; it is easily seen that wave-lengths corresponding to these overtones are, respectively, L and $\frac{2}{3}L$. From the preceding reasoning it follows that an open pipe yields the complete series of harmonics 1, 2, 3, 4, 5, 6, · · ·.

In a pipe closed at one end it is evident that there must always be a node at the stopped end. Hence, the different modes of vibration will be those represented in Fig. 363, having wave-lengths, respectively, $4L$, $\frac{4}{3}L$, $\frac{4}{5}L$. The fundamental of a stopped pipe is, accordingly, an octave lower than that of an open pipe, and its overtones are the odd notes in the harmonic series, their frequencies being the order of the numbers 1, 3, 5, 7, 9, · · ·. These relations between the fundamental and the overtones were discovered by Daniel Bernoulli, and are commonly known as Bernoulli's Laws.

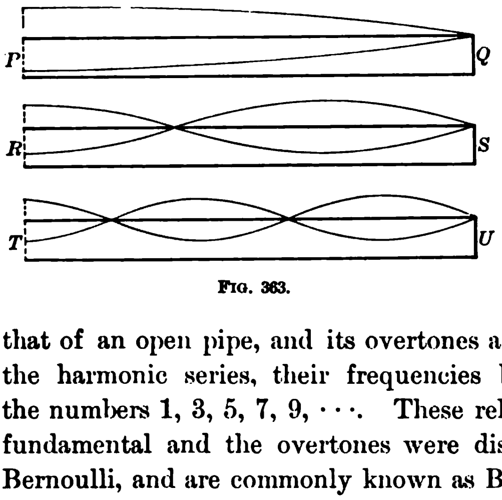


Fig. 363.

The assumption in this discussion that there is no variation of pressure at the open end of a pipe is not strictly true. It is, accordingly, found that the expressions for the wave-length require a correction, unless the length of the pipe is very great compared to its cross section. In the case of a cylindrical tube of radius R the actual wave-length is greater than the length of the tube by $0.6R$ for each open

end. For a square pipe this correction is roughly equal to the depth of the pipe.

The nature of the material of which the pipe is constructed may be neglected except when it is extremely thin.

It may be stated as a general law that if two bodies of air are geometrically similar, and similarly set in vibration, the pitches will be in the inverse ratio of their homologous linear dimensions.

499. Mouthpieces. — The mouthpieces by which the tones of musical pipes are generated may be grouped in three classes, according as the current of air is blown across a sharp edge, or through a reed, or between membranous tongues.

The mouthpiece used on the flute pipes of an organ is shown in Fig. 364.

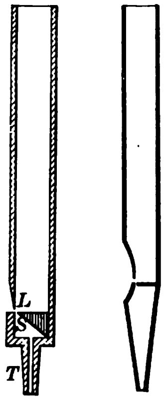


FIG. 364.

When a current of air is blown across the edge of a solid, it gives rise to a hissing sound, which may be regarded as due to a series of a great number of wave-lengths lying within ill-defined limits. Under certain circumstances, as in the rapid movement of a whip through the air, or in the whistling of the wind, the sound often approaches a definite pitch.

The experiments of Strouhall indicate that a current of air flowing past a cylindrical wire gives rise to a sound whose frequency varies directly as the velocity of the current and inversely as the diameter of the wire, but depends on nothing else, provided the temperature remain constant. It was also found that when the period of this note approached any of the free periods of the wire the intensity of the sound was greatly increased.

The excitation of flute pipes and all whistles is to be explained in the same way. When the limits of the periods of the disturbances, which are caused by the rush of the current of air across the edge of the mouthpiece, include the fundamental period of the pipe, it will resound in its lowest tone. If the velocity of the air current is now gradually increased, the frequency of the exciting disturbances will also increase, and as they come into the vicinity of the harmonics of the pipe it will speak in the pitch of its various overtones.

FIG. 365.

In the flute (Fig. 365) the *embouchure* is merely an oval hole, against the edge of which a current of air from the player's mouth is directed.

FIG. 366.

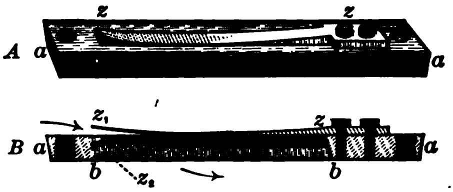


FIG. 367.

In the second class of mouthpieces the vibrations of the pipe are induced by a vibrating tongue or reed, which alternately opens and closes an aperture in which it is placed.

The reed most often used in organ pipes, and shown in Fig. 366, is known as a striking reed. The metal tongue l is slightly curved, so that it rolls itself over the aperture r, closing it gradually. The period of the reed, which is more or less coerced by that of the air in the pipe, may be altered and the pipe tuned by pushing down the wire d.

The free reed used in the accordion, harmonium, and rarely in pipes, is shown in Fig. 367. The tongue in this case swings through the aperture, interrupting the air current, whose direction is indicated by the arrows. In these instruments the free reed retains its own period unaltered.

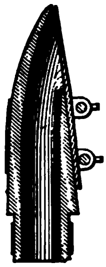


FIG. 368.

The mouthpiece of the clarinet is furnished with a striking reed made of thin cane (Fig. 368).

In the oboe and the bassoon the mouthpiece is in the form of a small wedge, the sides being formed of two tapering slips of cane, leaving a narrow slit at the apex (Fig. 369).

The period of these labial reed tongues is determined, for the most part, by that of the air column in the pipe, since

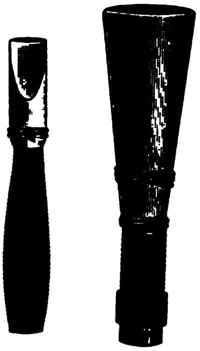


FIG. 369.


FIG. 370.

the instruments which employ them are capable of yielding a great number of notes.

An example of a mouthpiece formed by membranous tongues is shown in Fig. 370. It is made by cutting off a gutta-percha tube obliquely on both sides, and stretching two strips of india-rubber over the points thus formed, so as to

leave a narrow slit between them. If air be blown through the slit in either direction, the membranes will emit a note depending on the body of air in the tube.

There are but two examples of such membranous tongues which have any importance in music — the lips as used in playing the brass wind instruments, and the larynx in singing. In the trumpet, and instruments of that class, the mouthpiece has the form of a cup (Fig. 371), which when applied to the lips permits them to act as stretched membranes, determining the vibration of the air in the tube. The frequency of vibration depends in part upon the tension of the lips, and in part on the pressure of the air exerted by the performer.

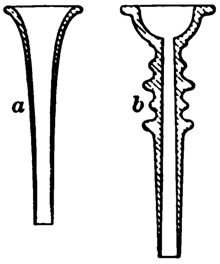


Fig. 371.

500. The Larynx. — The apparatus of the voice in man consists essentially of two elastic folds, known as the *vocal cords* c, c, Fig. 372, in which A represents a diagrammatic

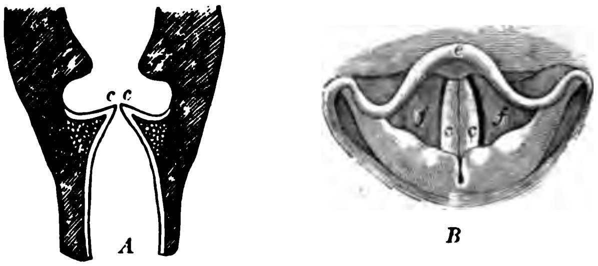


Fig. 372.

vertical section and B the position of the cords in tone production, as seen from above. These folds, when stretched across the head of the windpipe from front to back, leave a

small aperture between them, not unlike the model of Fig. 370. The frequency of vibration is determined by the form of the slit and the tension of the membrane, both of which may be altered at pleasure.

501. Kundt's Tubes. — The relative velocity of sound in gases and solids may be determined by the following simple method due to Kundt. The apparatus consists essentially of a tube of glass, *AB* (Fig. 373), fitted at one end with a movable stopper, *S*, and at the other with a tightly fitting cork, *B*, in which is fastened another tube or rod, *CD*, about a meter long and a centimeter in diameter. This rod is also terminated by a disc of cork which nearly fills the larger

Fig. 373.

tube. If the inner surface of *AC* be coated with a light dust, and a moist cloth be drawn along the rod *D* till it is set in longitudinal vibration, yielding its fundamental note, dust figures will be formed within the tube. By moving the stopper out or in, the dust may be made to collect at the nodes whenever the length of the air column is an exact multiple of the wave-length of the sound in air, which may then be taken as twice the distance between two nodes, *n*, *m*. But the wave-length in the rod is four times the distance *BD*. Hence, if V_a be the velocity of sound in air, and V_g be the velocity in glass,

$$V_g = \frac{CD}{mn} V_a. \tag{1}$$

By introducing any other gas into the tube it is evident that the velocity of sound in this gas might be found in terms of the velocity in air.

By the use of such a tube Kundt found that the velocity of sound relative to air was in steel 15.2, in carbon dioxide 0.8, in illuminating gas 1.6, and in hydrogen 3.56, numbers which are in substantial agreement with those calculated from formulas already given.

502. Resonance. — It is easy to show, experimentally, that when an elastic body, possessing a definite period, is subjected to a series of impulses, it will not be sensibly affected unless the period of these impulses corresponds to that of the body, when it may finally receive sufficient energy to set it in vibration with considerable amplitude. For example, let a tuning fork be struck and held over the mouth of a cylindrical vessel (Fig. 374) having a depth greater than the quarter wave-length of the note given by the fork. If water be gradually poured in, it will be found that at a certain level the sound of the fork will be strongly reinforced; but above or below this level the column of air within does not greatly affect the intensity of the sound.

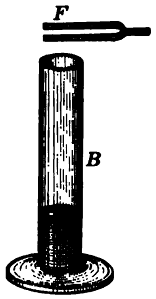


Fig. 374.

This phenomenon, in which one body sets in vibration another having a similar period, is termed *resonance*. The principle is extensively used in musical instruments to reinforce the sound. Thus the cords of all stringed instruments are attached to a sounding box with an air cavity, by the aid of which the energy is given out at a more rapid rate.

503. Resonators. — For the purpose of investigating the existence of partial tones in any note, Helmholtz devised the instrument which he called a *resonator*. It consisted of a globe of thin brass enclosing a mass of air possessing a defi-

nite period of its own. On one side there was a small open tube, b (Fig. 375), designed to fit into the meatus of the ear, and opposite it another opening, a, by which the enclosed body of air might receive vibrations from without. When one ear is stopped and such a resonator applied to the other, practically the only sound heard is that having the same pitch as the resonator. By using a series of such instruments, Helmholtz was first enabled to make a satisfactory analysis of a compound note into its partial tones.

Fig. 375.

504. The Vibrations of Strings. — The fundamental vibration of a musical string fixed at both ends may be regarded as identical with those giving rise to waves of the stationary type in a cord of unlimited length whose nodes correspond to the ends of the string. The period of a stationary wave is obviously the time it takes one of the progressive waves of which it is composed to move over a wave-length, *i.e.* twice the distance between the adjacent nodes. Hence, if l be the length of a string fastened at both ends and n its frequency, by equation 2, Art. 466,

$$n = \frac{v}{2l}, \tag{2}$$

or, substituting the value of v from equation 26, Art. 472,

$$n = \frac{1}{2l}\sqrt{\frac{T}{\mu}}. \tag{3}$$

The influence of the length, the tension, and the linear density on the pitch of strings is illustrated qualitatively in instruments of the violin type. Thus, as the length of the string is shortened by running the finger down the fin-

ger board, the pitch rises. The pitch is also raised by increasing the tension in the process of tuning. In the lower strings the period of the open string is lengthened by winding it with wire, thus increasing its linear density without affecting its elasticity.

For studying the relation between the length of a string and the pitch, the instrument shown in Fig. 376, and known as the sonometer, is a convenient one. It consists essentially of a wire stretched over two fixed bridges on a sounding-box. Between these a movable bridge is arranged to slide over a scale of equal parts so that the string may be stopped at any desired length. The string is set in vibration, either by plucking or bowing. If the lengths chosen are successively $\frac{8}{9}$, $\frac{4}{5}$, $\frac{3}{4}$, $\frac{2}{3}$, $\frac{3}{5}$, $\frac{8}{15}$, and $\frac{1}{2}$, the resulting notes will be found to be those of the major scale. By stopping the string at one-third or one-fourth of its length, the twelfth and the double octaves may be obtained. In these cases the existence of nodes and loops may be demonstrated by placing little paper riders on the string. If, for instance, in the case last mentioned the riders are placed at points an eighth of the string's length apart, and the string is bowed while the finger is placed against a point one-fourth of the length from one end, all the riders, except those at points dividing the string into four equal parts, will fall off, showing the existence of nodes at these points.

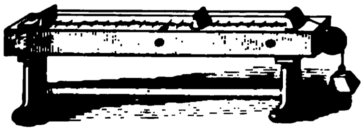


FIG. 376.

It is not difficult to recognize the presence of several overtones when the string is vibrating in its gravest mode. Thus, if the string be plucked at a distance of one-fourth of its length from the end, and then stopped at the center, the fundamental tone will cease, but the octave may still be

heard. Likewise, if the string be plucked at the center, and then stopped at one-third of its length, the twelfth will be recognized, and similarly with a large number of harmonics. It is, however, to be noticed that the series of these harmonics changes with the place where the string is struck, since this point can never be a node. Thus a string struck at the center will yield only the odd harmonics.

If a stretched wire be plucked near the extremity, the quality of the note is noticeably different, on account of the prominence given to some of the upper harmonics.

505. Transverse Vibrations of Rods. — The expression for the frequency of a vibrating rod may be shown to have the form

$$n = \frac{Ad}{l^2}\sqrt{\frac{E}{\rho}}, \tag{4}$$

where E is Young's Modulus, ρ the density, l the length of the rod, d its thickness in the direction of vibration, and A a constant depending on the mode of fastening and the number of nodes, but which cannot be determined completely without recourse to experiment. When a bar having both ends free vibrates so as to form two nodes, these are situated approximately at one-fifth of the length of the bar from the end.

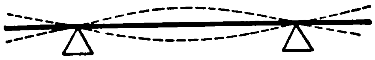


Fig. 377.

If a bar be laid on strings or sharp-edged corks at these points, as in Fig. 377, and struck with an ivory mallet, the note corresponding to the above mode of vibration will be obtained. In this way the dependence of the pitch on the length and thickness of the bar, stated in equation 4, may be verified. Thus it will be found:

1°. That two bars having the same length and thickness will have the same note, whatever their breadth.

2°. That if one of two bars having the same thickness has a length $\sqrt{\frac{1}{2}}$ times that of the other, its tone will be an octave higher.

3°. That if one of two bars having the same length is but half as thick as another, its tone will be an octave lower.

The frequency of the note of a bar vibrating in any other mode than the fundamental is quite different from that occurring in the case of strings. Beginning with the case of a rod fastened at one end, if its fundamental be denoted by $C_1 = (1.2)^2 n$, the overtones will be approximately,

C_1	$g^{\text{iv}}_{\sharp}$	d''	$-d'''$	$b^{\text{v}}_{\flat}$	$+f^{\text{vi}}$
$(1.2)^2 n$	$(3)^2 n$	$(5)^2 n$	$(7)^2 n$	$(9)^2 n$	$(11)^2 n$
1	$\cdot 6\frac{1}{4}$	$17\frac{1}{4}$	$34\frac{1}{4}$	$56\frac{1}{2}$	84

It will be observed that, beginning with the first overtone, the frequencies increase as the squares of the odd numbers, and hence form an inharmonic series.

In the case of a rod free at both ends, that is, vibrating with two nodes, as in Fig. 377, the series is the same as before, except that it begins on $(3^2)n$ as the fundamental. The series for a bar fixed at both ends is the same as for one free at both ends.

Vibrating rods are used, to a limited extent, for purposes of music in several ways. Four of their applications are enumerated below. In the xylophone the sonorous bodies are strips of wood supported at the nodal points and tuned to the notes of the scale through two or more octaves. In the Geneva music box the notes are produced by the mechanical plucking of the teeth of a metal comb, which are loaded so as to increase their period. The triangle is a bent bar occasionally used in the orchestra.

Chimes are sometimes made of long metal tubes instead of bells, as they can be more satisfactorily tuned.

The vibrations of reeds of organ pipes are largely coerced by the columns of air which they set in vibration.

506. Tuning Fork. — Another important example of the vibrating bars is the familiar tuning fork, commonly used as a standard of pitch on account of its freedom from all variations except those dependent upon temperature. If a fork, after being set in vibration, be held in the hand, it will continue in motion for a long time; but, since it gives up its energy to the air slowly, the sound will be very feeble. If, however, the stem of the fork be pressed against the top of a table, the latter experiences a series of forced vibrations, and the intensity of the sound is greatly increased, though it is necessarily of shorter duration, since the energy is given out at a more rapid rate.

The manner in which the fork is able to set the sounding board in vibration may be readily understood from Fig. 378. *A* shows how the nodes approach the center as the bar is bent more and more. *B* shows that, while the prongs of the fork are swinging back and forth, the bow where the stem is attached is moving up and down, but in an amplitude so small that contact with the board does not at once bring it to rest. For use in the lecture room, large tuning forks are mounted upon a wooden box, open at one or both ends, and of such length that the enclosed air has quite exactly the period of the fork, thus rendering the tone full and pure.

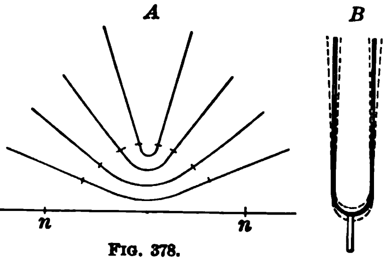


FIG. 378.

A rise of temperature diminishes the elasticity of the steel, and has the effect of lowering the pitch of the fork slightly. When the fork is set vibrating by bowing, the discordant overtones are very feeble and evanescent. The form of the wave in this case may be very exactly represented by the sine curve, but the quality of its tone, musically considered, lacks character.

507. Longitudinal Vibrations of Rods. — The existence of longitudinal vibrations in rods may be rendered visible by means of the apparatus shown in Fig. 379. An ivory ball suspended by two threads is arranged so as to rest against the end of a brass rod clamped at the center. If the outer end of the rod be stroked with a piece of rosined leather, it will emit an acute musical note, and if the rubbing be continued vigorously, the amplitude of the vibrations may be increased so as to impart a considerable impulse to the ball, which is driven away as often as it touches the end of the rod.

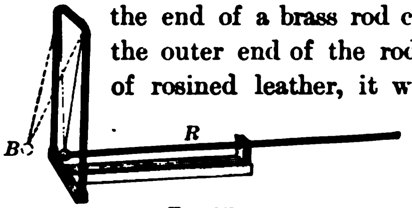


Fig. 379.

The frequencies of the different modes of vibration are related to each other as the natural series of numbers, as may be proved by the reasoning used for organ pipes (Art. 498).

Neither the torsional nor longitudinal vibrations of rods are used in music.

508. Vibrations of Plates. — The modes of vibration of plates were first investigated experimentally by Chladni, who found the position of the nodal lines by scattering fine sand uniformly over the plate held by a clamp, and then bowing the edge.

If a square plate, clamped at the center, be touched at the middle of one side while the edge is bowed at the corner, it will vibrate in its gravest mode, and the sand will collect along lines shown in Fig. 380.

The direction of motion of any segment changes on crossing a nodal line, as is indicated by the signs plus and minus.

If the plate be touched at the corner and bowed at the

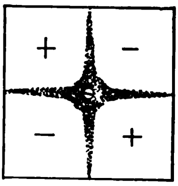


FIG. 380.

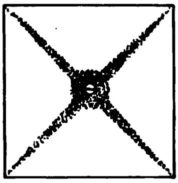


FIG. 381.

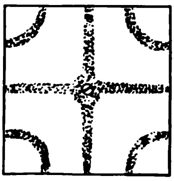


FIG. 382.

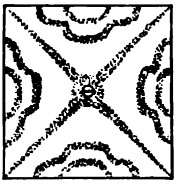


FIG. 383.

middle of a side, the nodal lines will now coincide with the diagonals (Fig. 381), and the note be a Fifth higher than that of the previous example.

Figs. 382 and 383 are two other examples, among a great variety of Chladni's figures, corresponding to the acuter modes of vibration.

The theoretical discussion of vibrating plates, and of membranes, which are somewhat analogous, is too difficult to be presented here.

Instruments like the cymbal and snare-drum have small musical value and are used solely to accent the rhythm of

military music. The kettledrum, used in the orchestra, permits of tuning, but simply that it shall not disturb the harmony of the other instruments.

509. Bells. — The modes of vibration of a bell resemble those of a plate in that the surface is divided by nodal lines, and the overtones form an inharmonic and often discordant series. The least number of segments in which a bell may vibrate is four. Thus, if the full circle (Fig. 384) represent the rim of the bell, the distortion resulting from a blow at some point on the side would be represented by the ellipse *MN*. By virtue of its elasticity it will immediately return not only to its original form, but will pass beyond, to a shape represented by *PQ*. If the bell is entirely symmetrical with respect to its axis of figure, the points *O*, *O*, *O*, *O* will remain relatively at rest, so that it will be possible to distinguish a difference of intensity of the sound opposite the points *M* and *O*. If, however, there be any want of symmetry in the figure, or homogeneity in the rim, these nodes may slowly revolve about the rim, producing the beats commonly heard when a large bell is struck.

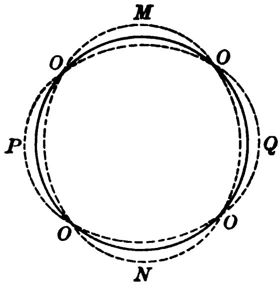


Fig. 384.

The form of a bell best adapted to suppress inharmonious partials can only be determined by trial. The one adopted for large bells, after many centuries of experiment, is similar to that shown


Fig. 385.

in Fig. 385. The tones vary with the thickness of the rim near the mouth, technically known as the sound bow. A bell may be tuned to a certain extent by chipping away a portion of the metal, but the pitch of the fundamental can be determined only by copying a pattern known to give the required note.

510. Lissajous's Optical Tuning. — The relation between the frequencies of two forks may be very conveniently studied by a method devised by Lissajous.

Let (Fig. 386) represent a fork bearing a small mirrror at the extremity of one prong vibrating vertically, and let *M* be a similar fork vibrating horizontally.

If a beam of light, *R*, be allowed to fall on the mirrors so as to form a small spot on the screen *S*, when the fork *L* is set vibrating alone, this spot, on account of the persistence

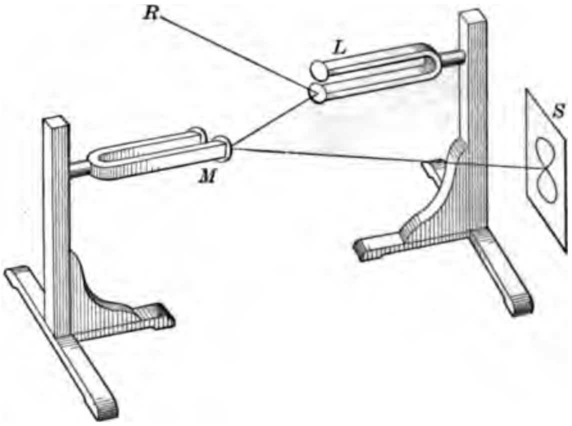


Fig. 386.

of the impression on the retina, will appear drawn out into a vertical line. Likewise, if *M* were set vibrating alone, the spot would trace a horizontal line. If both forks are set mov-

ing at the same time, their vibrations will be compounded and will produce the characteristic figures described in Art. 474. Thus the interval between the forks may be judged from the form of the figure, and the exactness of the tuning by its constancy of form.

It will be evident, from the explanation of Art. 474, that when the tuning is imperfect the change of the figure through a complete cycle means that one fork has gained or lost one vibration over the other, which corresponds to what the ear recognizes as a beat.

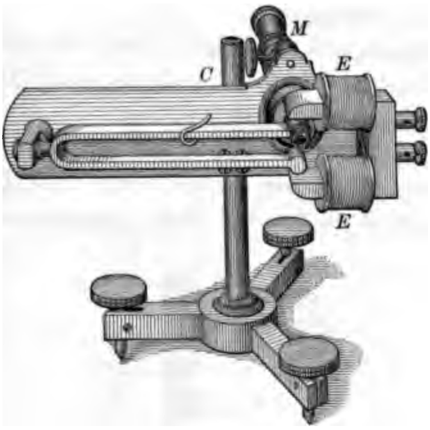


FIG. 387.

In practice the apparatus takes the more convenient form shown in Fig. 387. A microscope is arranged on a stand so that the ocular and tube shall remain fixed, while the objective, which is attached to a prong of a standard fork, is capable of vibrating in a vertical direction. Any small point observed through the microscope while this fork is sounding will consequently appear drawn out into a line. In order to make the vibrations of the fork continuous, the prongs are arranged between the poles of an electromagnet so that they automatically interrupt the current about the magnet with each vibration.

The fork to be examined is placed by the side of the first, so that the directions of their vibrations shall be at right angles. A bright point on the second fork, such as a globule

of mercury, observed by the aid of the vibration microscope, will, when both forks are set vibrating, appear to describe the Lissajous figures, by which the interval between the forks may be obtained with greater precision than is attainable with the ear.

511. Standard Pitch. — The starting point, or tuning note, for an orchestra is a', the note in the second space of the treble clef. In the practice of musicians its absolute value has varied considerably and undergone, on the whole, a marked rise. In 1751 Handel used a fork, $a' = 422.5$. In 1891 a prominent New York maker was tuning his pianos to a fork, $a' = 451.7$. The pitch at present receiving widest recognition is that adopted by the Paris Conservatoire, namely, $a' = 435$ at 15° C.

Several circumstances seem to have influenced the considerable rise of pitch noted. An error might be introduced in copying a standard fork by neglecting the influence of temperature. In making a new fork the prongs are left a little too long, so that the note is at first too grave. The pitch is then raised to unison by filing away a small amount of the metal. This manipulation will probably raise the temperature so that, even if the tuning be perfect when it leaves the workman's hand, after it has been allowed to cool, its note will be sharp by a small amount. It is thus possible that after a succession of copies a considerable error might arise. Historically the chief variation of orchestral pitch is to be ascribed to the makers or players of wind instruments, who, by raising the pitch, secure greater brilliancy of tone. It is also probable that this is true of piano makers.

512. Determination of Absolute Pitch. — The most important methods available for the determination of absolute pitch are, in order of increasing accuracy:

1°. *The Siren* (Art. 486). The difficulty of maintaining a constant velocity of rotation, and of judging a unison between a note of its screaming character with one of softer quality, introduces an uncertainty of at least 2 or 3 per cent, and frequently more.

2°. *The Vibroscope* (Art. 487). It is manifest that the free period of the fork will be influenced somewhat by the mass of the style, by its friction against the paper, and, when it is electrically driven, by the forced character of the vibrations. If, however, the period of a particular fork is carefully determined under certain definite conditions, from its graphical record the frequency of any other fork may be obtained from it with great accuracy by Lissajous's method of comparison.

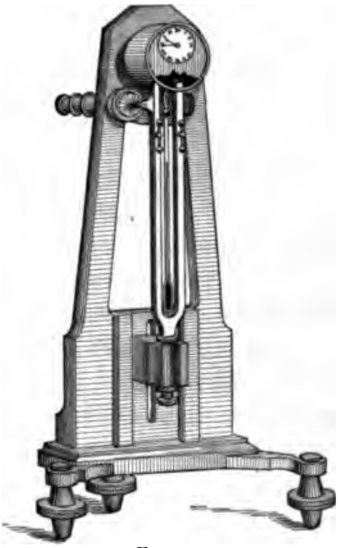


FIG. 388.

3°. *König's Clock-Fork.* In this method a large fork, having a frequency of about 64 vibrations per second, is made to serve as the isochronous element of a clock in place of the usual balance spring (Fig. 388). By observing the rate of this clock, the period of the fork may be determined to almost any degree of accuracy. Then, by means of the vibration microscope with which it is supplied, the period of any other fork may be obtained in the usual way.

EXAMPLES.

1. Find the frequency and wave-lengths of the first three tones emitted by an open pipe 67 cm. long when blown with air at 0°.

Ans. $n_1 = 247$ per sec., $\lambda_1 = 134$ cm.
$n_2 = 495$ per sec., $\lambda_2 = 67$ cm.
$n_3 = 742$ per sec., $\lambda_3 = 44.7$ cm.

2. What must be the length of a stopped pipe whose fundamental note should have a frequency of 520 vibrations per second?

Ans. 6.3 inches.

3. A rod 76 cm. long, when vibrating longitudinally in its fundamental mode, yields a note of 4520 vibrations per second. What is the velocity of the wave in it?

Ans. $6870 \frac{\text{met.}}{\text{sec.}}$

4. If an organ pipe be warmed from 16° to 127° C., how much will the note which it emits be affected?

Ans. About the interval from c to $d\sharp$.

5. What alteration in the pitch of a whistle in air will be made by blowing it with a gas whose density is $0.7173(10)^{-3} \frac{\text{gm.}}{\text{cc.}}$?

Ans. A Fourth.

6. A glass rod 73.2 cm. long, clamped at the center and rubbed with a rosined cloth, emits its fundamental note of 2780 vibrations per second. Calculate the velocity of sound in glass.

Ans. $4.07(10)^5$ cm. per sec.

7. 213 cm. of a uniform wire weigh 4.79 grams. What should be the note emitted by a piece of wire 47 cm. long, when stretched by a force of 24,800 grams' weight? *Ans.* $n = 350$ per sec.

8. If a vibrating string, when stretched with a weight of 16 pounds, emits the note a, what weight must be added to make it yield c'?

Ans. 7.04 pounds.

9. The length of the e string of a violin is 33 cm., and it has a mass of 0.125 gm. What is its tension when tuned to 640 vibrations per second? *Ans.* $0.676(10)^7$ dynes.

CHAPTER XXXIV.

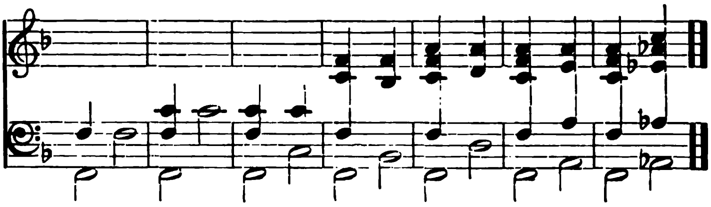
COMPOUND TONES.

513. Beats of Upper Partials. — In the case of simple tones beats are produced only when the pitch of the notes is nearly the same, but beats may be produced between compound tones whenever any of their partial tones nearly coincide, or the prime of one tone approaches the upper partial of another. Thus, if two notes differing by a seventh be sounded, the first overtone of the lower note will beat with the higher note, the number of beats being, as in the case of simple tones, the numerical difference of their frequencies.

If, now, the interval between the primes be increased, the number of beats will grow less and less until the octave is reached, when they entirely disappear, thus affording a very sensitive test of the accuracy of tuning.

The perfectly tuned intervals of the octave, twelfth, or fifth, being undisturbed by beats, are especially satisfactory to the ear. For this reason they are called *consonant* intervals, in distinction from the less harmonious, such as the second or seventh, which are termed *dissonant*.

The following examples in musical notation show the



Octave	Twelfth	Fifth	Fourth	Maj. Sixth	Maj. Third	Min. Third
$\frac{2}{1}$	$\frac{3}{1}$	$\frac{3}{2}$	$\frac{4}{3}$	$\frac{5}{3}$	$\frac{5}{4}$	$\frac{6}{5}$

coincidences and beating notes among the upper partials of the most important intervals, arranged in the order of their selection by the ear. The primes in each case are denoted by half notes and the overtones by quarter notes.

514. Dissonant Intervals. — The disagreeable impression produced on the ear by a dissonant interval is due, on the theory developed by Helmholtz, in every case to the presence of beats, which give an intermittent character to the sound. Why an intermittent stimulation of the auditory nerves should produce a more unpleasant sensation than an even stronger but continuous excitation may be better understood by comparing the analogous behavior of certain other nerves. In general, any considerable stimulation of a nerve deadens its sensibility, *i.e.* renders it less sensitive to succeeding excitation, but if left to itself the nerve quickly recovers its original sensitiveness. For instance, on passing from a dark room into a brilliantly illuminated one the light may at first appear so intense as to blind one, or to produce a painful sensation. But the continued action of the light, although partially excluded through the contraction of the pupil, soon diminishes the sensibility of the optic nerves so that the discomfort is relieved. If, however, the intensity of the light be intermittent, as in a flickering gas flame, where periods of illumination are followed by intervals during which the nerves of the eye recover, the irritation produced will evidently be much greater than that from a steady source. The same holds true of the nerves of touch, so that the scraping of the finger-nail, or the light brushing of a feather over the skin, is more annoying than the same pressure continuously applied.

The effect of an intermittent or beating note on the ear may thus be regarded as comparable with that of a flicker-

ing gas flame on the eye, or a tickling feather on the skin, producing a more irritating and unpleasant sensation than would be occasioned by a continuous tone.

Helmholtz found that, as the number of beats was increased, a maximum of discomfort was reached at about 33 per second, for notes of medium pitch, after which, on account of the persistence of impression, the intermittent character is less easily distinguished, just as the eye is unable to differentiate a succession of impressions exceeding a certain frequency. A glowing stick, for instance, when whirled with even a moderate velocity, appears as a continuous ring of fire.

515. Development of the Scale. — The fact that the consonant intervals of the octave, fifth, and fourth, could be obtained by using similar strings, whose lengths were in the ratios $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, respectively, was discovered by Pythagoras four centuries before the Christian era, and modern physicists have known since the middle of the eighteenth century that the intervals of the scale could be expressed by ratios between the first five numbers, or their simple multiples; but previous to the investigations of Helmholtz no one could give any other explanation for these remarkable relations than that the human mind had a peculiar pleasure in simple ratios.

In the light of Helmholtz's theory of consonant intervals, it is easy to see that the selection of these particular notes of the scale by all civilized nations has been determined chiefly by the natural relationships which exist between the keynote and the other tones.

If after sounding one note the octave is struck, the ear hears again nothing but what was present in the lower compound note, though the intensities may be different. In sounding the fundamental and the twelfth, there is a repeti-

tion in the upper note of some of the partials of the first note. Proceeding in this manner, selecting those notes whose natural relationship is such that they have at least one partial tone the same in each series of overtones, the following notes are obtained belonging to a single octave and arranged in the order of their relationship to the tonic *c*:

$$\begin{array}{ccccccc} c & c' & g & f & a & e & eb \\ \frac{1}{1} & \frac{2}{1} & \frac{3}{2} & \frac{4}{3} & \frac{5}{3} & \frac{5}{4} & \frac{6}{5} \end{array}$$

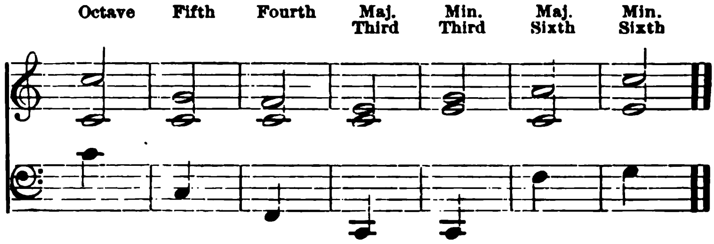
Neglecting the minor third, these notes form all of the major scale except two notes, which are inserted to divide the rather long intervals from *c* to *e* and from *a* to *c'*. The value of $b = \frac{15}{8}$ may be regarded as having been selected either to form the major third above *g*, or, what is more probable, so as to form a leading note to the octave *c'*. The position of $d = \frac{9}{8}$ was probably chosen to form the fifth of *G* in the octave below.

516. Combinational Tones. — Two notes when sounded together may produce tones whose frequencies are, respectively, the sum and difference of the frequencies of the components. The difference tones were discovered by Sorge, and independently by Tartini, after whom they are frequently called the *tones of Tartini.*

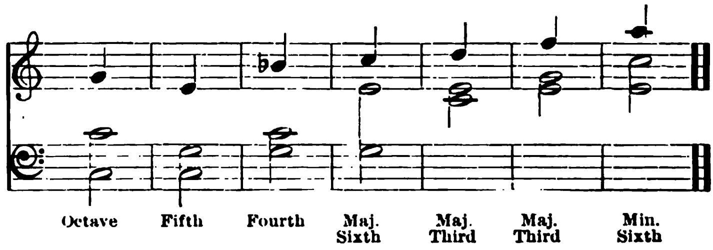
The summation tones were first predicted by Helmholtz, from a theory which he had developed to account for difference tones, and afterwards found by experiment.

Combinational tones may be formed between the overtones of two compound notes as well as between the primes, and even in some cases between the combinational tones themselves. The resultant tones of the first order, arising from the usual harmonic intervals, are as follows, the gen-

erating tones being denoted by half notes and the resultant tones by quarter notes:



Difference Tones.



Summation Tones.

The summation tones are much more difficult to perceive than the difference tones, which may be readily obtained from the siren, harmonium, or two tuning forks, when any of these instruments are sounded loudly.

Since combinational tones are observed only when the generating tones have considerable intensity, they are, in the theory of Helmholtz, regarded as the resultant of a secondary system of waves arising from a type of vibration which may no longer be treated as harmonic for considerable amplitudes.

Assuming the force of restitution to be a quadratic function of the displacement, Helmholtz showed that secondary systems of waves would be produced whose frequencies were, respectively, the sum and difference of those of the harmonic primaries.

517. Vibration of Bowed Strings. — By the aid of the vibration microscope, Helmholtz was able to determine the precise character of the vibration of bowed strings. The vibrations of the standard fork being performed in a horizontal line and those of a string very nearly in unison with it, in a vertical line, the series of figures shown in Fig. 389 was obtained.

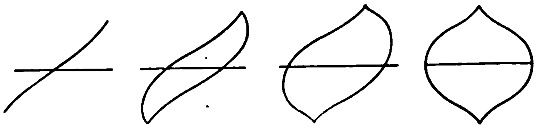


FIG. 389.

As any of Lissajous's figures may be regarded as the projection of a curve which has been plotted in terms of the time and the displacement and then wrapped about a cylinder (Art. 474), it is evident that the curve representing the displacement as a function of the time in the figures of Fig. 389 may be obtained by simply developing the cylinder. The result thus obtained for a point near the middle of the string, when the bow bites well, is shown at *A* (Fig. 390). If the point observed is nearer the end, the curve takes the form

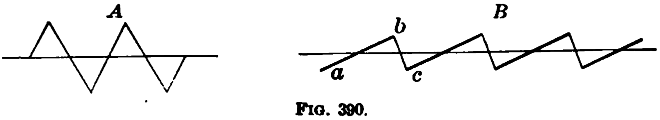


FIG. 390.

shown at *B* (Fig. 390). These figures enable us to form a clear conception of the process of bowing. When the rosined strands of the bow are applied to the string, they adhere and the string is displaced a short distance by the motion of the bow, since for small stresses rosin behaves like a viscous fluid. But as soon as the force of restitution in the string exceeds a certain value, the string will break away from the bow, — for rosin behaves like a brittle solid to stresses

exceeding a definite amount, — and fly back to the opposite side of its position of rest, where it is again gripped by the bow, to be again drawn aside. The string thus receives a series of impulses which have sensibly the period of its own free vibration.

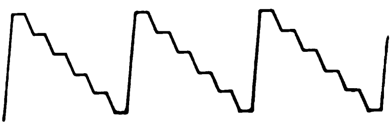


FIG. 391.

Since Fig. 390 defines the form of the wave, it is possible to determine from these lines the presence and intensity of the upper partials.

The fundamental form of the vibrations of a violin string is nearly independent of the place where it is bowed, differing in this respect greatly from plucked or struck strings.

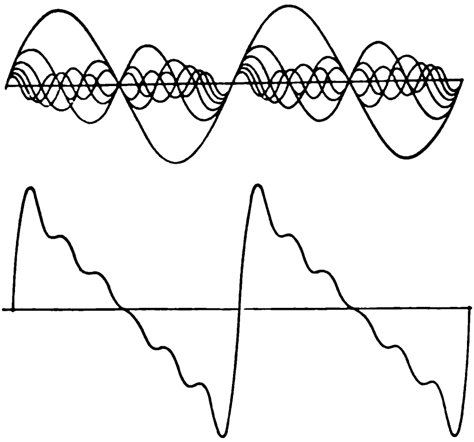


FIG. 392.

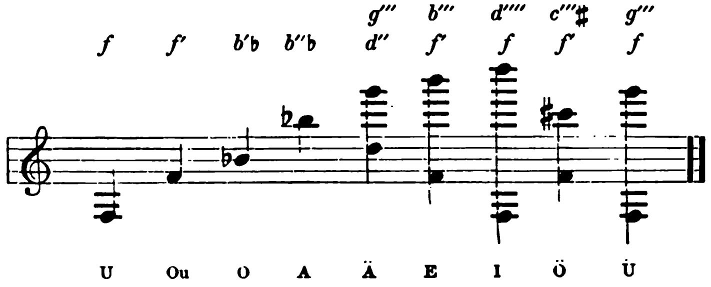
When the point of bowing was not near the end of the string, Helmholtz found that little crumples were introduced on the lines of the vibrational figure, as in Fig. 391.

If a fundamental wave, with its five upper partials varying in amplitude as the wave-length, be added geometrically, the resulting curve (Fig. 392) is seen to bear a very close resemblance to Fig. 391. Helmholtz concludes from his figures that, in a violin string which speaks well, all the upper partials are, in general, present, but with an intensity which diminishes as the square of their frequency increases.

518. Quality of Vocal Sounds. — The different qualities of vocal sounds are determined chiefly by the form of the glottis and the tension of the vocal cords, but the intensity of the overtones is considerably modified by the resonance which takes place in the cavity of the mouth. In order to produce a smooth, pure tone, it would seem to be necessary that the edges of the vocal cords be straight and capable of close and perfect alinement without striking. The quality of screaming and rasping voices would seem to be due to imperfect fulfillment of these conditions, permitting the vocal cords to overlap and strike together. At least these sounds resemble those produced by striking reeds.

Helmholtz is of the opinion that in the speaking voice the vocal cords act as striking tongues. Hoarseness in colds is due in part to the relaxation of the vocal cords incident to their inflammation, and in part to flakes of mucus which rest in the glottis, disturbing the motion of the cords.

By the aid of resonators it has been shown that six or eight partials are commonly present, though in different degrees of intensity, depending on the form of the cavity of the mouth. The fact that this cavity is tuned to different pitches may be easily recognized in the quality of prolonged whispered vowels. By means of the resonance excited by tuning forks, Helmholtz found that this cavity for different (German) vowels was tuned as follows:



519. Synthesis of Vowel Sounds. — By means of a series of tuning forks with resonators which could be adjusted so as to produce any required intensity of each note, Helmholtz was able to imitate a number of the vowel sounds. His results are exhibited in the following table, in which the upper row shows the harmonic series of forks used, and the common musical marks of loudness, *pp*, *p*, *mf*, *f*, *ff*, indicate the intensity of each tone used to produce the vowel sound designated at the left.

	1 $b\flat$	2 $b'\flat$	3 f''	4 $b''\flat$	5 d'''	6 f''	7 $a'''\flat$	8 $b'''\flat$
U	*f*							
O	*mf*	*f*	*p*					
A	*mf*	*mf*	*mf*	*f*	*f*			
Ä		*f*	*f*	*p*	*ff*	*ff*		
E	*mf*	*mf*				*ff*	*ff*	*ff*

It was further found possible, with the apparatus mentioned, to imitate the quality of the sounds produced by several sorts of organ pipes.

From these and similar experiments Helmholtz drew the important conclusion that the quality of a compound tone depends solely on the number and intensity of its partial tones, and not upon their difference of phase.

520. Musical Quality of Sounds.—The nearly simple tones obtained from tuning forks and wide-mouthed stopped pipes, although free from roughness, are not satisfactory musical sounds on account of their dull or colorless quality. Compound tones, consisting of a prime accompanied by a series of moderately loud overtones up to the sixth, such as those of open organ pipes, the French horn, or the softer tones of the human voice, are described, musically, as full and rich.

When only the harmonics, corresponding to the odd series of numbers, are present, as in stopped organ pipes, the tones are described as hollow if there are but few overtones present. When a large number of upper partials are present, such sounds are termed nasal. When overtones above the seventh are very distinct, the quality of the notes is piercing and rough on account of the dissonances which they form with one another. Such tones are often called wiry, reedy, or brassy, after the nature of the instruments which produce them.

521. Anatomy of the Ear. — The apparatus of hearing in man consists of three well-marked portions, distinguished as the *external*, the *middle*, and the *internal ear*. The external ear consists of the auricle (*pinna*), the familiar appendage at

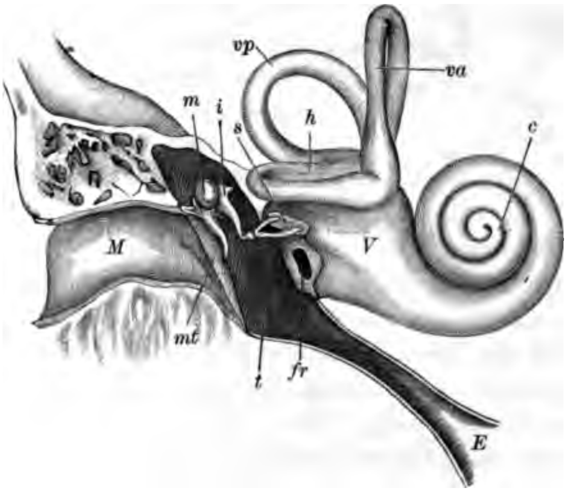


FIG. 393.

M, meatus; *m*, malleus; *i*, incus; *s*, stapes; *mt*, membrana tympani; *t*, tympanum; *fr*, fenestra rotunda; *c*, cochlea; *V*, vestibulum; *h*, horizontal semicircular canal; *va*, vertical anterior semicircular canal; *vp*, vertical posterior semicircular canal; *E*, Eustachian tube.

the side of the head, and the auditory canal (*meatus*), *M* (Fig. 393, which shows a section through the right ear). In the lower animals the auricle may assist in locating the direction of a source of sound, but in man its influence on hearing is insignificant. The meatus is closed by a membrane (*membrana tympani*), *mt*, commonly called the ear-drum, which is stretched across the tube with its outer surface slightly concave.

The middle ear is an air cavity (*tympanum*), *t*, completely shut off from the external ear by the drum, and from the inner ear by a bony wall. This wall is pierced by two membrane-covered holes, known respectively as the oval window (*fenestra ovalis*) and the round window (*fenestra rotunda*). The tympanic cavity contains a chain of three small bones, known as the hammer (*malleus*), *m*, the anvil (*incus*), *i*, and the stirrup (*stapes*), *s*. The articulation of these ossicles is more plainly shown in Fig. 394. The handle *Mm* of the malleus is attached to the inner side of the tympanic membrane a little below the center, and the head of the stapes to the membrane of the oval window.

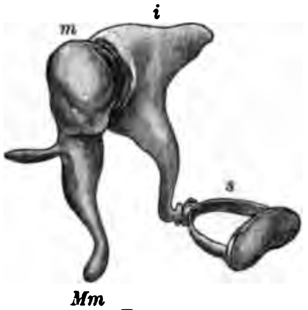


FIG. 394.

The middle ear communicates with the external air by means of the Eustachian tube *E* (Fig. 393), extending into the upper part of the throat, but generally closed except in the act of swallowing. By closing the mouth and nostrils and compressing the air in the pharyngeal cavity, the tube may be forced open and the pressure in the tympanum increased, an effect which is easily recognized by a characteristic crackling sensation in the ear. Upon swallowing, however, the pressure will be again equalized.

The internal ear, or labyrinth, is a partly membranous and partly bony organ embedded in very dense bone and filled with a watery fluid. It consists of three parts: the vestibule, *v* (Fig. 393); the semicircular canals, *h*, *vp*, *va*; and the cochlea, *c*. The outer wall of the vestibule is pierced by the two membrane-covered windows already mentioned. The lower, or round window, permits a slight expression of the fluid whenever the membrane of the oval window is pushed in by the stirrup.

Opening out of the vestibule, and continuous with it, is the cochlea, consisting of a spiral tapered tube, so named from its resemblance to a snail shell. From the central column of bone a partition, *Sp* (Fig. 395), extends outward toward the center of the cochlear canal, where it is met by two membranes, the *membrana vestibularis*, *mv*, and the *membrana basilaris*, *mb*. The upper gallery thus formed ends in the vestibule and is called the *scala vestibuli*. The lower one, extending to the round window, is known as the *scala tympani*, and the central one the *ductus cochlearis*.

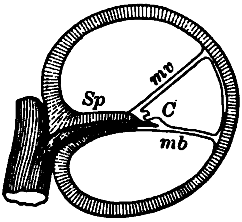


FIG. 395.

A portion of the auditory nerves, rising through the axis of the cochlea, pass over the bony partition and terminate on a multitude of rod-like fibers at *C*, called, from their discoverer, the rods of Corti. These rods form a double series, about 3000 in number, increasing somewhat in size as they approach the vertex of the cochlea. The fibers of each pair are joined at the top and spread at the bottom, where they rest on the *membrana basilaris*, so as to form a sort of arch over it, as shown in Fig. 396.

The epithelium, near the terminals of another portion of the auditory nerves which pass into the vestibule, is covered

with a number of stiff elastic hairs, extremely well adapted for moving sympathetically with the fluid in which they are immersed.

In the vicinity of these surfaces there are also found a number of calcareous concretions, known as *otoliths*, designed

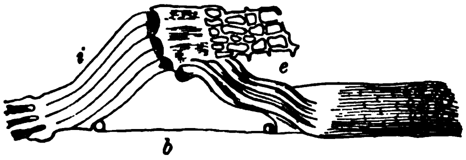


FIG. 396.

apparently to produce a mechanical irritation in the nerves of the surface upon which they rest, whenever they are agitated.

522. Functions of the Several Parts of the Ear. — When a train of sound-waves falls on the ear, the agitation of the air near the drum will set this membrane in a similar vibration. By means of the ossicles of the middle ear these vibrations are reduced in amplitude and transmitted to the fluid of the labyrinth, and thence to the ciliated cells of the vestibule and the basilar membrane with the superposed organ of Corti.

The function assigned to the last-named parts is one in accord with the well-known ability of the ear to distinguish a complex wave, not as the eye does, by its peculiar form, *i.e.* a succession of varying total displacements, but by resolving it into its component harmonic vibrations. The only mechanical analogy of the analysis of a compound periodic motion into simple harmonic vibrations is that afforded by the phenomena of sympathetic vibration. Thus, for instance, if all the dampers be raised from the strings of a piano and one of the vowels be spoken (or, better, sung) close to the

sounding board, only those strings will be set in vibration which have the same frequency as the simple tones which were contained in the given note. In this case the echo, or resonance, of the strings is a sufficiently perfect copy of the original sound to permit the original vowel to be easily recognized.

In the case of the ear it is not possible to ascertain which parts, if any, vibrate sympathetically with each note; but the whole construction of the basilar membrane and the arches of Corti seem so well adapted for performing independent vibrations, that it appears most probable that it is by the aid of this apparatus that we are enabled to resolve any musical sound into its simple components.

The tuning is apparently determined by the breadth at different points of the *membrana basilaris*, which increases in width from about 0.041 mm. at the base of the spiral to 0.495 mm. at its apex. As already pointed out, there is a corresponding, though not proportionate, increase in the Corti's arches as they approach the vertex of the spiral.

Under these circumstances, those portions of the membrane in unison with the higher notes must be looked for near the round window, and those corresponding with the deeper near the vertex.

The ciliae of the vestibule may be regarded as assisting in the perception of squeaking, hissing, or crackling sounds.

The semicircular canals *h*, *va*, *vp* (Fig. 393) are so arranged that one lies essentially in a horizontal plane, another parallel to the median plane, and the third in a vertical right and left plane. They seem to have no connection with hearing, but rather to be the organ of another sense, *viz.* that of equilibration. At least the destruction of one or more of the semicircular canals in the lower animals is accompanied by remarkable disturbances in their ability to preserve equilibrium.

523. Duration of Aural Impression. — When the rapidity of beats between two notes exceeds a certain number, different for different pitches, the ear no longer recognizes the sound as an intermittent one. By examination of the smallest number of beats which one note makes with another, in order that the sensation shall be continuous and the interval an essentially consonant one, Mayer has found the following values for the duration of different tones :

NOTE.	BEATS.	DURATION.
		sec.
$C = 64$	16	$\frac{1}{16} = 0.0625$
$c = 128$	26	0.0384
$c' = 256$	47	0.0212
$g' = 384$	60	0.0167
$c'' = 512$	78	0.0128
$e'' = 640$	90	0.0111
$g'' = 768$	109	0.0091
$c''' = 1024$	135	0.0074

524. Resonance of External Canal. — According to the researches of Helmholtz, the air in the external auditory canal plays the part of a resonator for sounds in the region between $e'''' = 2640$ and $g'''' = 3168$, strongly reinforcing these tones. To some ears these notes when loud are almost painful.

It is for the same reason that the stridulant note of the cricket seems relatively so intense. The sound is much weakened by applying a small tube to the ear, so as to lower the pitch of the resonance chamber.

It is a notable fact that the human voice is especially rich in those overtones to which the ear is peculiarly sensitive.

CHAPTER XXXV.

MUSICAL INSTRUMENTS.

525. Organ. — The church organ is a great assemblage of metal and wood pipes, sounded by means of air supplied under pressure by a bellows and admitted through valves operated by keys under the control of a single performer. The pipes are usually arranged in four departments, each a more or less complete instrument with a keyboard of its own, and known respectively as the great organ, the choir organ, the swell organ, and the pedal organ. The keys of the first three are arranged in as many banks or manuals, one above the other in the order mentioned. The last is operated by a row of foot-keys or pedals, whence its name.

Each department of the organ is composed of a number of sets of pipes or "stops" having the same quality through a certain range of pitch. The different octaves on the organ are usually distinguished by the length of the open pipe used to give the prime. Thus, the octave C to B is called the eight-foot octave.

The peculiar quality of the various stops is determined in part by the kind of mouthpiece (Art. 499), though more especially by the form and dimensions of the pipe, which reinforce certain overtones and extinguish others. In the stops called *mixtures* the note is produced by sounding together a number of pipes, each tuned to the individual tones of the harmonic series.

Several forms of organ pipes are shown in Fig. 397.

The method of tuning organ pipes varies with the nature of the pipe. Closed wood and metal pipes are tuned by means

of a sliding stopper. Open metal pipes are usually flattened by expanding the upper end of the pipe by means of a cone

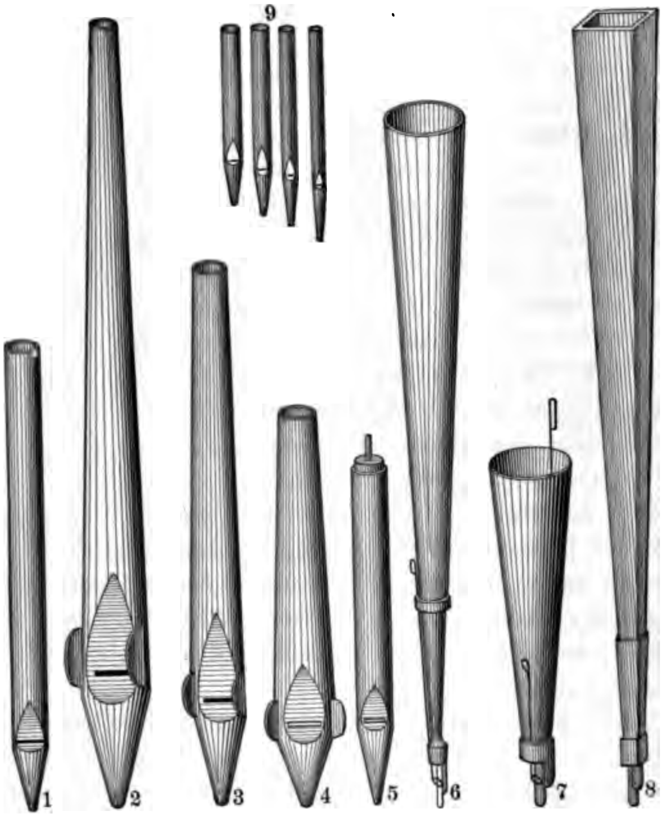


FIG. 397.

1. Principal (4 foot). 2. Spitz-flöte (8 and 4 foot). 3. Twelfth (3 foot). 4. Cornet. 5. Flute (8 and 4 foot). 6. Trumpet (8 and 4 foot). 7. *Vox humana* (8 foot). 8. Bombarde, or double reed (16 and 8 foot). 9. Mixture (4 ranks).

driven into it, and sharpened by contracting this orifice. Delicate stops are often tuned by means of ears (Fig. 397), which may be bent over the mouth. Open pipes are some-

times tuned by means of a sheet of lead projecting over the top, and sometimes by means of a slide which opens a slit at the top (Fig. 361). All reed pipes are tuned by shortening or lengthening the tongue of the reed by means of a wire resting against it (see Fig. 366).

The pitch of organ pipes is often seriously impaired by the changes of temperature, which affect the velocity of sound within the pipes (Art. 494). The disturbances from this cause are further aggravated by the fact that the reed pipes are not affected to the same extent as the flute pipes.

526. Violin. — The most important of bowed instruments is the violin. It consists of four gut strings, the lowest covered with silvered copper wire, stretched over a bridge on a wooden chest (Fig. 398).

FIG. 398.

The strings are tuned to fifths as follows: g, d', a', e'', by means of pegs over which they pass. The intermediate and higher notes are obtained by stopping the strings with the tips of the fingers, which are pressed against the finger board.

The compass is about three octaves and a half, the high notes being obtained by touching the string lightly so as to make it yield the harmonics.

The vibrational form of the string discussed (Art. 517) is not exactly that of the wave which reaches the ear, for the string does not itself readily communicate its vibrations to the air. This transfer of energy is effected by the sonorous body of the instrument, which naturally favors some of the partials at the expense of others.

527. Viola. — The viola is similar to the violin, except that it is somewhat larger. It has four strings of gut, the

lower two of which are covered with silvered copper wire. They are tuned to fifths as follows: c, g, d', a'.

528. Violoncello. — The violoncello is a still larger instrument of about the same shape with two plain and two wire-covered gut strings. They are tuned to fifths as follows: C, G, d, a.

529. Double Bass. — The largest instrument of the violin class, called the double bass, is usually furnished with four strings tuned to fourths, thus: E_1, A_1, D, G. It is an indispensable part of the orchestra.

530. Flute. — The flute (Fig. 399) is a tube of wood or silver with a cylindrical bore stopped at one end with a plug by which the instrument may be tuned. The mouthpiece (Fig. 365) is simply an oval hole near the closed end, against the thin edge of which the performer directs his breath. The sides of the tube are pierced with a number of holes designed to be opened and closed by the fingers or keys. When all the holes are closed and a moderate pressure is used, the instrument yields its fundamental note, which in the concert flute is usually d'. By opening the other holes in order, the acoustic length of the tube is shortened, and the succession of notes up to $c''\sharp$ may be obtained. By repeating this process, using a greater wind pressure, the next higher scale may be obtained, and in like manner a third, making the compass of the flute about three octaves. The *piccolo* is a similar instrument, but pitched an octave above the flute. The *fife* is a simpler variety of the piccolo.

Fig. 399.

531. Oboe. — The oboe (Fig. 400) consists of a wooden conical tube slightly flaring at the lower extremity, and furnished with a peculiar mouthpiece, which has already been described in Art. 499. Its compass is usually from b to f''. Like the flute, its harmonics are those of an open pipe. It is a difficult instrument to play, but it has a rich tone, which admits of wide variation in intensity, and hence is capable of every variety of expression. The oboe usually gives the pitch to the strings in the orchestra.

Fig. 400.

532. Bassoon. — The bassoon (Fig. 401) is the natural bass of the oboe. It has a similar mouthpiece, but the tube, about ninety inches in length, is bent back upon itself. Its compass is from $B\flat$ to c'', and its harmonics are those of an open pipe.

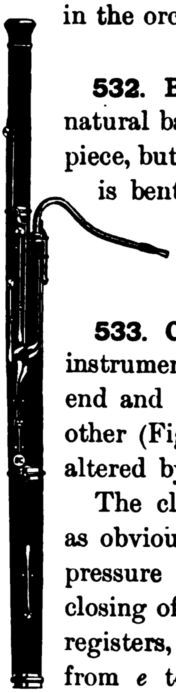


Fig. 401.

533. Clarinet. — The clarinet (Fig. 402) is a wood instrument of cylindrical bore, ending in a bell at one end and a mouthpiece with a single beating reed at the other (Fig. 368). The effective length of the tube is altered by opening holes in its side.

The clarinet yields only the odd series of harmonics, as obviously should be the case from the variations of pressure at the mouthpiece incident to the opening and closing of the reed. This instrument has two principal registers, differing by a twelfth, the lower one extending from e to $b'\flat$, and the upper, obtained by opening a special hole, from b' to $c''\sharp$. By cross fingering a still higher octave may be obtained. Clarinets are made in several keys to avoid the difficulties of execution which arise in trying to play all music on one instrument.

FIG. 402.

534. Saxophone. — The saxophone (Fig. 403) is an instrument resembling the clarinet in many respects, but differing from it in being made of brass with a bent conical tube. Its usual compass is from b' to f''', but it is made in several other keys. The saxophone, though much esteemed in military music, is but little used in the orchestra.

FIG. 403.

535. Trumpet. — The trumpet is the most ancient of wind instruments, and may be regarded as the prototype of all instruments with cupped mouthpieces. It consists of a long brass tube, tapering near the outer end into a bell (Fig. 404). Its notes are those of an open tube, about as in the diagram below.

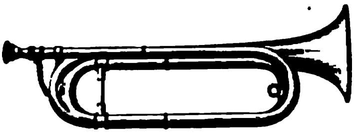
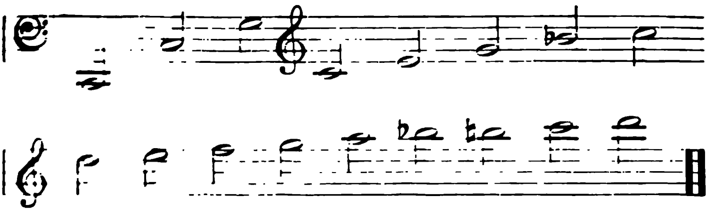


FIG. 404.

On account of the difficulty in striking it, the lowest note is not used.

The trumpet used in orchestral music is furnished either with a slide like the trombone (Art. 537), or with valves like the cornet (Art. 538).



536. French Horn. — The French horn (Fig. 405) consists of a tube about seventeen feet in length, terminating in a wide bell, and yielding the same series of harmonics as the trumpet. It possesses, however, a more mellow quality of tone, determined in part by the shape of the mouthpiece (Fig. 371 *a*), and in part by the shape of the bell. Its pitch may be altered by the use of auxiliary tubes, called crooks, by which the acoustic length of the instrument may be increased or diminished. The missing notes of its scale were formerly interpolated by partially stopping the bell with the hand, but the modern orchestral instrument has been given a fixed chromatic scale by the introduction of a slide and valves.

FIG. 405.

537. Trombone. — The trombone (Fig. 406) is a modified form of trumpet, in which the length of the tube is capable

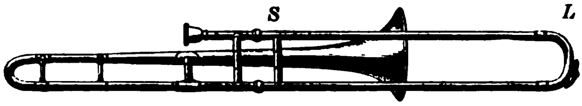


FIG. 406.

of continuous variation, throughout a considerable length, by means of the slide *SL*. It is the only one of the wind instruments which possesses a perfect scale; its compass is from about *E* to $b''\flat$.

538. Cornet. — The cornet (Fig. 407) is essentially a trumpet which has been modified by the introduction of

three slides through which the wind may be made to pass, by depressing the valves, or pistons, *a*, *b*, *c*. The effective length of the tube is thus changed by amounts necessary to flatten the open tones by one, two, or three semitones. The pitch of the cornet may be altered by changing the length of the tube between the mouthpiece and the instrument. Cornets are made in several sizes and keys, but their usual compass is from c to g'''.

FIG. 407.

539. Saxhorn. — The saxhorn, or tuba, is a name given to a family of bass wind instruments of the general type of that shown in Fig. 408. They have the usual harmonics of an open pipe, the chromatics and intermediate tones being obtained by the use of pistons, as in the cornet.


FIG. 408.

Saxhorns have a compass of about three octaves, and being made in various sizes and keys, furnish all the necessary foundation tones for the military band. They possess the notable advantage of a uniform fingering. The lowest of all the contra-bombardon, on account of its weight and size, is coiled in circular form so as to be carried on the shoulders of the performer.

PART V.—LIGHT.

CHAPTER XXXVI.

NATURE AND PROPAGATION OF LIGHT.

540. Meaning of the Word Light.—The word light is commonly used in two distinct senses: 1°, to designate the sensation which is characteristic of the organ of vision; and, 2°, as the name for the usual cause of that sensation. This double meaning of the word would cause little inconvenience if there were always a definite relation between the sensation and its cause; but this is far from being the case, since, as will be shown hereafter, the same sensation may be produced in a variety of ways. In order to avoid confusion, the first use of the word has been confined to a single chapter—the Forty-Fourth. In that chapter light is treated as a sensation alone. Elsewhere it is regarded as a phenomenon of wave motion entirely apart from the sense organ which reveals its existence to us.

By white light are meant such waves as are emitted by a solid body at a very high temperature, as, for example, the incandescent lime in the calcium light. Any other collection of waves, even though indistinguishable by the unassisted eye from these, is not white. Likewise, by yellow light, green light, etc., are meant those waves of definite length which will excite in the normal retina the sensations yellow, green, etc.

541. Rectilinear Propagation. — In a homogeneous medium, light originating at a luminous point may be regarded as propagated in straight lines, so that the shadow of any obstacle in the path of the light may be determined to the first approximation by the locus of tangents drawn through the luminous point and the bounding line of the opaque body, according to the principles of geometry.

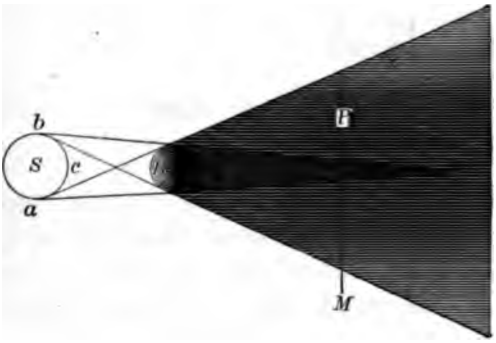


Fig. 409.

If the source of light is a luminous body, *S* (Fig. 409), the shadow of an opaque body, *B*, on a screen at *M* will not be sharply defined, for the reason that a point, such as *P*, for instance, though screened from the portion of the luminous disc *ac*, is still illuminated by the portion *bc*. The region of total shadow is called the *umbra*, and that of partial and varying shadow the *penumbra*.

Even when the source of light may be regarded as a point, the edge of the shadow is not perfectly distinct, for the reason that points near the edge of the obstacle themselves become new centers of disturbance, giving rise to a phenomenon known as diffraction (see Chapter XXXIX).

542. Shadow Pictures. — If light proceeding from any luminous object be allowed to pass through a minute hole in an opaque screen, and fall upon a white sheet in a dark room, an inverted picture of the object will be formed.

Thus, in Fig. 410, light proceeding from AB, and passing through the hole H in straight lines, will illuminate minute areas at ab on the screen in proportion to the brightness of each source, so that there will appear on the screen a figure reproducing the outlines, color, and varying brightness of the original object, but with a distinctness which diminishes as the size of the hole is increased.

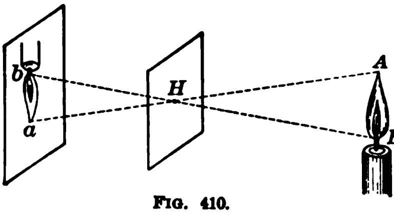


FIG. 410.

543. Velocity of Light.—*Römer's Method.* The first definite knowledge concerning the velocity with which light is propagated was deduced by Römer, a Danish astronomer, in 1675, from consideration of certain peculiarities which had been observed in a series of eclipses of a satellite of Jupiter. The mean time between successive eclipses of this satellite, the first, was about forty-two and one-half hours; but when the earth was approaching Jupiter this period was slightly shortened, and when it was receding the period was lengthened. Thus, suppose S (Fig. 411) represents the sun, E, the earth, and J, Jupiter. If the first observation was made when Jupiter was in opposition, that is at J in the figure, it was found that when Jupiter came to conjunction at J', the eclipse did not appear to occur till about 1000 seconds later than its calculated time of immersion, whence Römer concluded

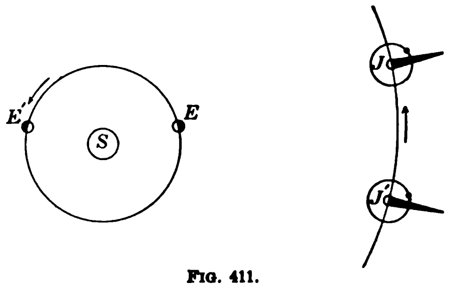


FIG. 411.

that this time must have been required for the light to traverse the difference of path between $J'E'$ and JE, that is, the diameter of the earth's orbit. Assuming this distance to be 186,000,000 miles, the velocity of light is found to be about 186,000 miles per second.

Bradley's Method. In 1727 Bradley, while seeking to determine the parallax of the fixed stars, discovered that they appeared to describe a small orbit about their mean position in the period of a year.

The explanation of this phenomenon, given by its discoverer, leads to an expression by which the velocity of light may be determined. To derive this, suppose first that P (Fig. 412) is a point moving with a velocity represented by PQ above a sheet of paper, AB, which is moved in a direction at right angles to PQ with a velocity RQ. The motion of the point with respect to the paper, as shown by its trace, will be PR. In a similar manner, if RQ represent the velocity of the earth in its orbit, and PQ the velocity of light coming from a star in a direction perpendicular to the plane of the ecliptic, the light will appear to come in the direction PR; that is, the star will appear displaced from its actual position by the angle RPQ. Calling u the velocity of the earth in its orbit, D the distance of the earth from the sun, α the angle of aberration, and T the period, one year, the velocity of light is

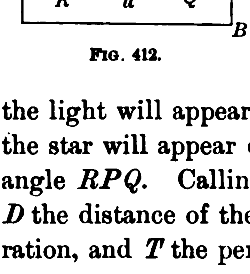


Fig. 412.

$$v = u \cot \alpha = \frac{2\pi D}{T} \cot \alpha. \tag{1}$$

Taking $D = 93(10)^6$ miles, and $\alpha = 20.5''$, the value of v is found to be practically the same as that quoted above. If the

velocity of light be assumed as determined by one of the succeeding methods, equation 1 may be used to calculate D. The distance of the sun so determined is regarded as the most accurate yet known.

Fizeau's Method. The first purely experimental determination of the velocity of light was made by Fizeau in 1849. Light from a bright source, S (Fig. 413), was allowed to fall upon a piece of unsilvered glass, G, whence it was reflected, passing between two teeth of a cogwheel, W, to a mirror several miles distant. From this it was again reflected to the first station, where, passing through the same gap in the wheel and the glass G, it entered the eye of the observer. The wheel was then set revolving at a gradually increasing velocity until the light was cut off from the eye, an occurrence

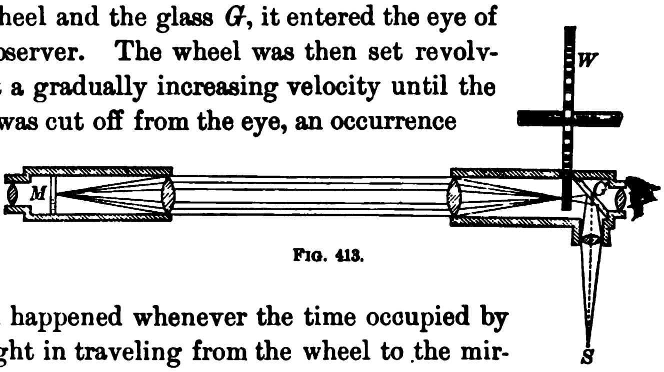


FIG. 413.

which happened whenever the time occupied by the light in traveling from the wheel to the mirror and back was equal to the time it took the tooth to move forward its own width and block the gap through which the light passed. The speed of the wheel being observed at this instant, the velocity of light can be calculated at once when the distance between the stations is known. Thus, suppose that this distance is d, and that the wheel must make n revolutions per second in order to produce an eclipse of the light. Calling m the number of teeth in the wheel, and assuming that the width of a tooth is equal to the width of a space, the velocity of light is

$$v = 2dmn. \tag{2}$$

The velocity found by Fizeau, 314,000,000 meters per second, is somewhat in excess of the true value.

Foucault's Method. The most accurate determinations of the velocity of light have been obtained by an arrangement of apparatus originally devised by Foucault and shown in Fig. 414. S is a slit by which sunlight may be admitted, G is a piece of plane glass, L a lens, R a plane mirror which may be revolved rapidly in the direction shown, and M a fixed concave mirror. When the mirror R is at rest, light entering at S falls on R and is reflected to M, by which it is sent back over its previous course to G. The portion reflected from this surface is then observed by the eye placed at S'. If, however, the mirror R be given a rapid rotation, then, while the light travels from R to M and back again, the mirror will have turned through a small angle, so that the light on its second reflection from R will depart in a direction slightly different from RS, and may be observed at some point, as S''. Thus, from a knowledge of the distances SR, RM, $S'S''$, and the angular velocity of the mirror, the velocity of light may be calculated. The most recent determinations of this velocity by Michelson and by Newcomb indicate a value in a vacuum of $2.9986(10)^{10}$ cm./sec.

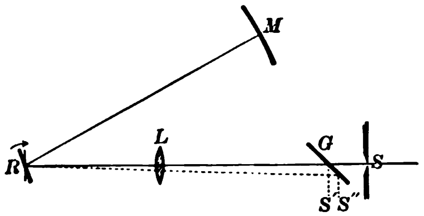


FIG. 414.

544. Theories Concerning the Nature of Light.—Two theories of the nature of light have been held, known respectively as the wave theory and the emission theory. In the last quarter of the seventeenth century Huyghens published a paper in which he explained the familiar phenomena of

light by means of waves in a medium which pervades all space and is called the luminiferous ether. The reasoning was so convincing, the explanations so simple, and the experiments supporting his views so apt, that it could hardly have failed to receive at an early day the universal acceptance which it now commands, except for the influence of a single philosopher, then living, who was greater than Huyghens himself. A few years earlier, in 1669, Newton had commenced his labors in the field of optics, by which, largely on account of the fame and authority which he had won in the domain of mechanics and astronomy, he established a theory of light which remained almost unquestioned for nearly a century and a half. Newton supposed light to consist in extremely small particles of matter projected from shining bodies with enormous velocities. It has since appeared that this hypothesis was not only less fruitful than that of Huyghens, but even within the comparatively limited range of optical phenomena known to Newton and his contemporaries it was less probable.

It was left to Fresnel to establish the wave theory by a remarkable series of papers, extending from 1815 to his death in 1827, upon a foundation which leaves no room for doubt.

There is one experiment which must be regarded as a crucial test between the two theories, though historically it came too late to be of any service in overthrowing the emission theory, and that is the velocity of light in different media. According to the theory developed by Newton, light should move with a greater velocity in an optically dense medium, such as water, than in a rare one, such as air, while on the wave theory the velocity should be greater in air than in water. The apparatus of Fig. 414 was designed primarily for the determination of this question, and by it Foucault showed that the velocity of light was considerably greater in

air than in water, though he did not make any estimate of the ratio. Its value has been recently determined in this manner by Michelson, who found it to be 1.330 for yellow light — a number in substantial agreement with that found by refraction (Art. 555).

Very much remains to be learned as to the character of the ether and its relations to the forms of ponderable matter; but it now seems certain that light-waves involve an electrical displacement at right angles to the direction of propagation, as first suggested by Maxwell. It also appears that waves of all lengths traverse the ether with the same velocity, for otherwise the reappearance of Jupiter's moons after an eclipse would be accompanied by changing color, resulting from the successive arrival of waves of different length.

545. Light a Wave Phenomenon. — To prove that any phenomenon is due to wave motion it is sufficient to show, 1°, that it is periodic, and, 2°, that it is propagated with a finite velocity. As light has already been shown to have a velocity, it only remains to prove its periodicity. This may be done by the aid of a lens of small curvature pressed against a piece of plate glass (Fig. 415).

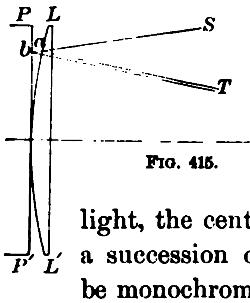


Fig. 415.

On looking at this by reflected light, the center will appear dark, but surrounded by a succession of bright and dark rings, if the light be monochromatic, or colored, if the incident light be white. As there is clearly nothing periodic about the apparatus, it follows that the periodicity must inhere in the light, which is accordingly shown to be a wave phenomenon.

The explanation of the rings is as follows: a portion of the light falling upon the surface LL' at a is reflected in the

direction T, and another portion which has passed through to the surface PP' will be reflected from the point b also in the direction T. Now, as the waves in the line bT have had to traverse a path longer than that of the waves in aT by double the distance ab, it is evident that they must be somewhat behind those of Ta in phase. Also, since the reflection at b takes place in the rarer medium, there will be an additional loss of half a wave-length in those waves reflected from the plane surface. At the center of the lens the difference of phase is just a half wave-length, and the two systems destroy each other.

Proceeding outward from the center, the difference of phase increases until it becomes a whole wave-length, when the two systems conspire, forming a bright ring. A little farther out extinction will again occur, the bright and dark spaces being thus repeated for a considerable distance from the center when light of a single wave-length is used. When the incident light is white, the rings are colored, but only about seven alternations can be traced. If the air film be illuminated successively by the light in the solar spectrum from red to blue, the diameter of the rings will gradually diminish, indicating that the red waves are longer than the blue, in something like the ratio of two to one. By a measurement of the diameter of one of the rings and the curvature of the lens, the wave-length of any color of the spectrum might be determined. Thus, suppose that the radius of the surface LL' (Fig. 415) is R, and the distance of the m^{th} bright ring from the center of the lens is d. The distance ab for perpendicular incidence may be taken as $\frac{d^2}{2R}$ (Art. 24). Now, the condition that there shall be a bright spot at this point requires that this double distance shall be equal to an odd number of half wave-lengths, or,

$\lambda =$
(

Calculation made in this v
a wave-length of about $\frac{1}{4500}$

EXAM

1. What is the length of the cone
cast by the sun, and the diameter of
of the moon? The distance of the s
its diameter as 109 radii of the ear
moon as $3.84(10)^{10}$ cm.

Ans. Length, $1.38(10)^{1}$

2. In an experiment by Fizeau on t
between stations was 8663 meters. W
teeth made 12.6 revolutions per second
value does this give for the velocity?

CHAPTER XXXVII.

REFLECTION AND REFRACTION.

546. Modification of Waves at the Boundary of Two Media. —The laws governing refraction and reflection of plane waves, at a boundary plane separating two media, have already been derived in Art. 480. The discussion will now be extended to the modification of waves at any surface. Suppose that

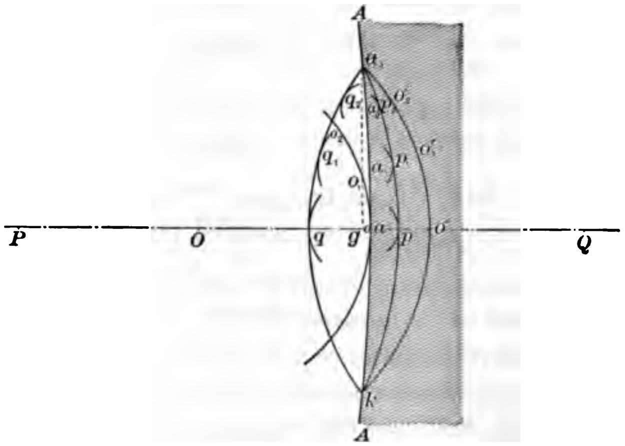


FIG. 416.

AA (Fig. 416) is the interface between two media, and that O is a source of light-waves in the medium in which they are propagated with the greater velocity. When the wave front reaches the point a, this point may be regarded as a center of disturbance from which, by Huyghens's Principle, a secondary wave would, after a brief time, have spread backward to q, and, if the medium were everywhere the same, for-

ward to o'. In the case assumed it will only have reached some point, p, nearer to a. A moment after o reaches a the point o_1 in the wave front will arrive at a_1, which in turn becomes a new center of disturbance. The secondary wave from this point will be found passing through q_1 in the first medium, and through p_1 in the second, at the instant that the wavelet from o is at p and q, the distance a_1q_1 being taken less than o_1o_1' by the distance o_1a_1, and a_1p_1 in the same ratio to a_1o_1' that ap is to ao'. In a similar manner the secondary waves from any other point in the interface may be constructed. The envelope of all the little spheres through $qq_1q_2 \cdots$ is the front of the reflected wave, and, similarly, a surface through $pp_1p_2 \cdots$ is the front of the refracted wave. This construction is obviously perfectly general and may be applied to all cases of reflection and refraction.

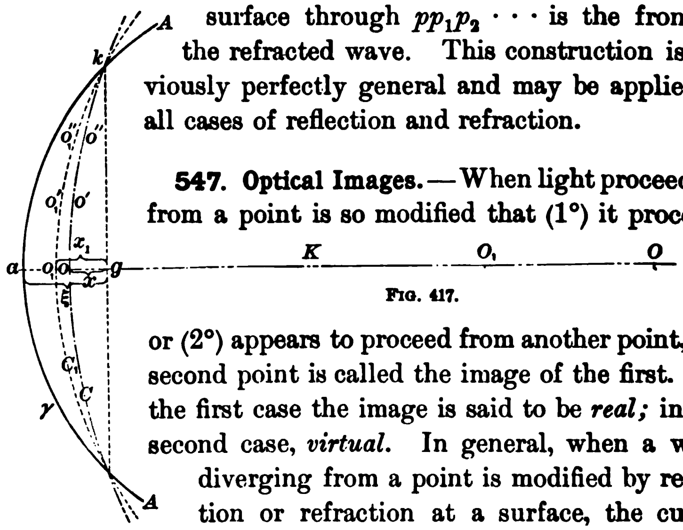


FIG. 417.

547. Optical Images.—When light proceeding from a point is so modified that (1°) it proceeds or (2°) appears to proceed from another point, the second point is called the image of the first. In the first case the image is said to be *real;* in the second case, *virtual.* In general, when a wave diverging from a point is modified by reflection or refraction at a surface, the curvature of the wave is so changed that it is no longer spherical and no image will be formed. If, however, the boundary between the two media, A, A (Fig. 416), is spherical, those portions of the new wave fronts within a few degrees of the axis qo' may be regarded as approximately spherical with centers P and Q, which are, respectively, the

reflected and refracted images of O. In the case represented they are, according to the definition, both virtual. Under the limitations mentioned, a formula may be derived which contains implicitly the whole theory of mirrors and lenses. To find this, let the curvature $\frac{1}{R}$, at any point of a surface, be called positive when the center of curvature is in the same direction from the surface as the propagation of the light. In Fig. 417 let

$\gamma =$ curvature of the interface,
$C =$ " " " incident wave front $o, o', o'', \cdots$
$C_1 =$ " " " modified " " o_1, o_1', o_1'',
$y =$ the common half chord gk,
$\xi =$ " sagitta ag,
$x =$ " " og,
$x_1 =$ " " o_1g.

Then, by geometry,

$$\xi = \tfrac{1}{2} y^2\gamma; \; x = \tfrac{1}{2} y^2 C; \; x_1 = \tfrac{1}{2} y^2 C_1. \tag{1}$$

If $\rho =$ the ratio of the velocity of light in the medium to the right of AA to that in the medium to the left,

$$\xi - x_1 = \rho\,(\xi - x), \tag{2}$$

whence, by substitution of the values from equation 1,

$$\gamma - C_1 = \rho\,(\gamma - C), \text{ or}$$

$$C_1 = \rho C + (1 - \rho)\,\gamma, \tag{3}$$

which is the equation sought.

548. Reflection from a Plane Surface. — When light is reflected from a plane surface, it is to be observed that its velocity is reversed in direction, and that the curvature of a plane is zero. Substituting $\rho = -1$ and $\gamma = 0$ in equation 3,

$$C_1 = - C; \tag{4}$$

that is to say, the light appears to proceed from a point as far behind the mirror as the object is in front. The image is, therefore, virtual. Regarding any figure, *ABCD* (Fig. 418), as an assemblage of luminous points, it follows that its image in a plane mirror, *MN*, is an equal figure, *A'B'C'D'*, symmetrically placed with respect to the mirror, but differing from the object in this respect: that, seen from the mirror, *B* is on the right of *A*; while in the image, *B'* is on the left of *A'*. This change is called *perversion*, and the image a *perverted* image.

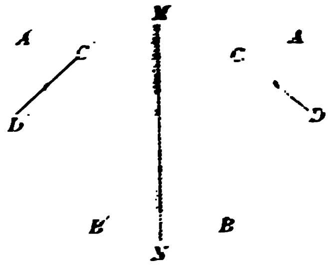


Fig. 418.

It follows, from the results just obtained, that, to the perception of an eye placed in front of an unbounded polished plane, like a mirror, the half of the universe behind the mirror is annihilated and replaced by a perverted copy of the half in front of it. If the mirror is not unbounded, the space behind may still be regarded as occupied by a perverted copy of what lies before, and the bounded mirror as a window through which we can look into this space. From these simple considerations it is obvious that the smallest mirror by which a person can see his whole figure is one having half the length and width of the observer, and that quite independently of his distance from the mirror. If two mirrors, *DE*, *EF*, be placed at right angles to each other, as in Fig. 419, one of the mirrors forms a perverted

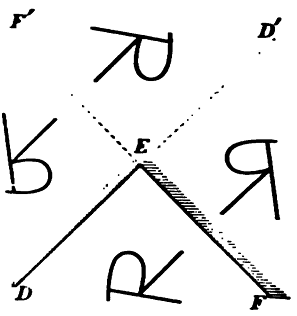


Fig. 419.

image of the region between the two, and the other a perverted image of the first image, so that in the region $F'ED'$ there is formed an unperverted image of the region between the mirrors. To a person familiar only with his features as they appear in a single mirror, any want of symmetry, as in the arrangement of the teeth for example, becomes very striking when seen in an unperverted image.

549. Reflection from a Spherical Surface. — To apply formula 3 to reflection from a spherical surface it is sufficient to set $\rho = -1$ when the curvature of the reflected wave becomes

$$C_1 = -C + 2\gamma, \tag{5}$$

which signifies that the curvature of the incident wave is increased by twice the curvature of the mirror after reversal of its direction. Calling the radii of the incident wave, the reflected wave, and the surface of the mirror, respectively, r, r_1, and R, equation 5 may be written

$$\frac{1}{r_1} = \frac{2}{R} - \frac{1}{r}. \tag{6}$$

For the degree of approximation assumed in proving the formula r and r_1 may be taken as the distance of the luminous point and its image from the mirror.

Case 1. *Concave Mirrors.* Making γ negative, then $C_1 = -2\gamma$ when $C = 0$; that is, plane waves are modified so that they converge to a point half-way between the mirror and its center of curvature. This point has received the name of the *principal focus* of the mirror. As C changes from 0 to -2γ, C_1 changes from -2γ to 0; that is to say, every point on the axis between infinity and the principal focus has a real image between the principal focus and infinity. At the center of curvature the image and the object coin-

cide. If C is negative and numerically greater than 2γ, C_1 is positive, or, in other words, the image of any point nearer than the principal focus is virtual.

If the waves which fall on the mirror are converging, that is, if C is positive, C_1 will always be negative and numerically greater than 2γ, or real images will be formed between the mirror and the principal focus.

Case 2. *Convex Mirrors.* The formula for a convex mirror is found by using the positive value of γ in equation 5, thus

$$C_1 = - C + 2\gamma.$$

The images of all luminous points will be virtual, lying between the mirror and the principal focus. Converging waves, having a curvature between ∞ and 2γ, will be brought to a focus in front of the mirror; but waves of a smaller curvature will have a virtual focus.

550. Images and Magnification. — Let K (Fig. 420) be the center of curvature of the mirror, O a luminous point in the axis, O_1 its image, and A a point very close to O. Draw a line through A and K. The image of A will be at A_1, very close to O_1. Construct also the incident and reflected wave surfaces C and C_1 belonging to A and A_1 and passing through V. Then, by the law of reflection, C_1 and C make equal angles with γ at the point V, or $AVO = A_1VO_1$, whence

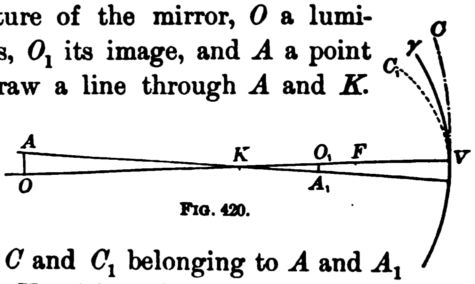


FIG. 420.

$$\frac{\overline{AO}}{\overline{A_1O_1}} = \frac{\overline{OV}}{\overline{O_1V}};$$

that is to say, the linear dimensions of an object and its image are in the same ratio as their distances from the vertex.

It is obvious from Fig. 420 that the real image of any object is inverted, but that the virtual image is erect.

551. Refraction at a Plane Surface. — For refraction at a plane surface, equation 3 becomes

$$C_1 = \rho C,$$

whence the image is always virtual.

If the light-waves originate in the optically denser medium, as at O (Fig. 421), they will appear to proceed from O_1, where

$$O_1 V = \frac{1}{n} OV.$$

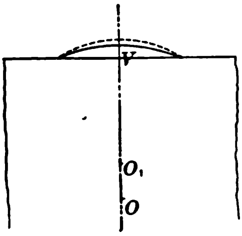


FIG. 421.

Thus, since for water $n = \frac{4}{3}$, the apparent depth of a lake will be but three-fourths its real depth.

The law of refraction for oblique incidence, equation 47, Art. 480,

$$\sin i = n \sin r,$$

may also be proved by a direct application of Huyghens's Principle, as in the construction of Fig. 416.

The case in which incident waves approach the surface at a large angle in the denser medium presents a singularity known as *total reflection.* Thus, as the angle of incidence OVN (Fig. 422) is increased, the angle of refraction O_1VN_1 increases also until it becomes a right angle, for which the angle of incidence is PVN. For a greater angle of incidence there can be no emergence, and the light is totally reflected, as at QV, VQ_1. The angle PVN, at which emergence ceases, is called the *critical angle.*

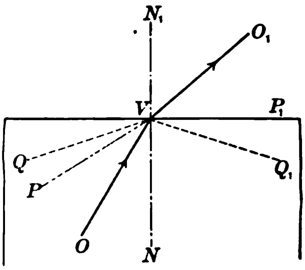


FIG. 422.

By the law of refraction,

$$n \sin PVN = \sin \frac{\pi}{2},$$

or the sine of the critical angle is equal to the reciprocal of the index of refraction.

For water the critical angle is about 48.5°. In different specimens of glass it varies between 38° and 41°.

552. Refraction through Two Plane Surfaces. — If the second medium has another boundary parallel to the first at a distance, t, from it, then the waves which appear to diverge from the first image will have their curvature increased by refraction through A_2B_2, in the ratio of n to 1; that is, in Fig. 423,

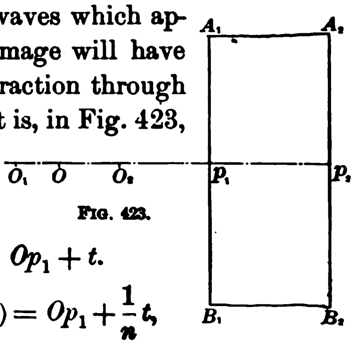


FIG. 423.

$$O_2p_2 = \frac{1}{n} O_1p_2;$$

but $$O_1p_2 = O_1p_1 + t = n \cdot Op_1 + t.$$

(7) $$\therefore\ O_2p_2 = \frac{1}{n}(n \cdot Op_1 + t) = Op_1 + \frac{1}{n}t,$$

which may be written

$$(Op_1 + t) - \left(1 - \frac{1}{n}\right)t;$$

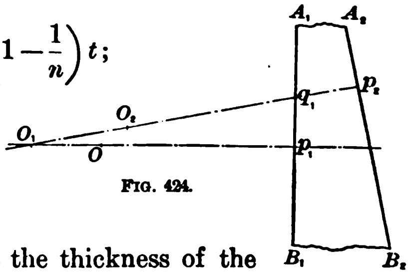


FIG. 424.

that is to say, a point seen through a layer of some denser medium will appear nearer than its true distance by $\left[1 - \frac{1}{n}\right]$ times the thickness of the plate. When the second boundary A_2B_2 (Fig. 424) is not parallel to A_1B_1, the first refraction produces an image, O_1, on the perpendicular Op_1, and the second refraction, through A_2B_2, an image, O_2, on O_1p_2, which is nearer A_2B_2 than O by

$\left[1-\frac{1}{n}\right]$th of q_1p_2, nearly. Thus it appears that the effect of this system, known in optics as a prism, is to bring the image nearer and displace it toward the thin edge, if, as has been assumed, the waves travel more slowly in the second medium than in the first.

553. Deviation by a Prism. — In the case when the incident waves are plane let the course of the light through the prism be $UVWX$ (Fig. 425), and call the angle of incidence

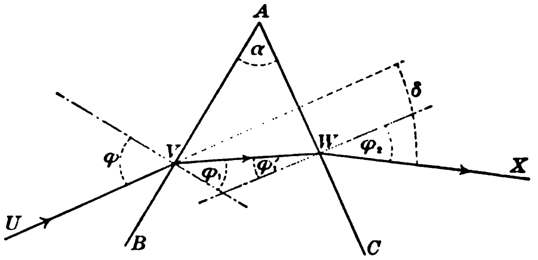


Fig. 425.

at the first surface ϕ, the angle of refraction ϕ_1, the angle of incidence on the second surface ϕ_1', and the corresponding angle of refraction ϕ_2. The difference in direction between WX and UV is called the deviation. Let it be denoted by δ; then

$$
\begin{aligned}
\delta &= \phi - \phi_1 + \phi_2 - \phi_1' \\
&= \phi + \phi_2 - (\phi_1 + \phi_1') \\
&= \phi + \phi_2 - \alpha. \qquad (8)
\end{aligned}
$$

It is obvious from the figure that a decrease of ϕ_1 corresponds to an increase of ϕ_1' and by consequence of ϕ_2, hence, the deviation is an increasing function of the index of refraction.

When $\phi_1' > \sin^{-1}\left(\frac{1}{n}\right)$ there will be total reflection.

Also, since $\phi_1' = \alpha - \phi_1$, this condition may be written $\phi_1 < \alpha - \sin^{-1}\left(\frac{1}{n}\right)$. The different deviation of waves of different wave-lengths will be discussed in Art. 580.

554. Minimum Deviation.—If in Fig. 425 the light be supposed to traverse the prism in the reverse direction $XWVU$, it is obvious that the deviation would be the same as before, whence it follows that for every angle of incidence within the limit of emergence there is another value, ϕ_2, for which the deviation will be the same. Suppose that i and i' are such a pair of values, i being less than i', and let the angle of incidence be gradually changed from i to i'. During this process δ will also vary, but since at the end it resumes its initial value it must have passed through a maximum or minimum. If, now, other pairs of values of i and i' be taken, making the difference $i' - i$ smaller and smaller, it is obvious that the stationary value of δ occurring in the change from i to i' must correspond to the case where the difference vanishes, that is, when, in Fig. 425, $\phi = \phi_2$. That δ is a minimum and not a maximum may be shown most simply by an experiment in which the direction of the incident light is varied, though it may be also proved analytically.

When $\phi = \phi_2$, we have also $\phi_1 = \phi_1' = \frac{\alpha}{2}$, and $\delta = 2\phi - \alpha$, whence $\phi = \frac{\alpha + \delta}{2}$, and

$$n = \frac{\sin\phi}{\sin\phi_1} = \frac{\sin\left(\frac{\alpha+\delta}{2}\right)}{\sin\frac{\alpha}{2}}. \tag{9}$$

This result is of great practical importance, since it enables the index of refraction to be calculated from the measured values of the minimum deviation and the angle of the prism.

555. Indices of Refraction. — The relative velocity of light in two media is always taken in such sense that the ratio shall be greater than unity.

If
$$n_{01}=\frac{v_0}{v_1},$$
$$n_{12}=\frac{v_1}{v_2},$$
$$\cdots\cdots\cdots$$
$$n_{k-1,k}=\frac{v_{k-1}}{v_k},$$
then, obviously,
$$n_{0k}=n_{01}\cdot n_{12}\cdots n_{k-1,k}.$$

The ratio of the velocity of light in a vacuum to the velocity in any medium is called the absolute index of refraction of that medium. As the velocity of light, in general, diminishes with the wave-length, a statement of the index of refraction of any substance beyond two or three significant figures requires the specification of the wave-length also.

The following table gives the approximate value for several important substances.

Water	1.33	Canada Balsam	1.54
Sea Water	1.34	Rock Salt	1.54
Alcohol	1.37	Glass, Crown	1.51 to 1.54
Turpentine	1.47	" Flint	1.56 to 1.78
Linseed Oil	1.48	Diamond	2.47 to 2.75
Carbon Disulphide	1.68	Lead Chromate	2.5 to 2.97
Air	1.00029		

556. Refraction through Lenses. — Pieces of transparent substances such as glass, quartz, etc., bounded by polished curved surfaces, are called *lenses*. Those which are thicker

in the center than at the edge are called positive lenses, and those which have the edges thicker than the center are called negative lenses. The former may produce a real image of an object, but the second never can. Some of the common forms of both types are shown in Fig. 426.

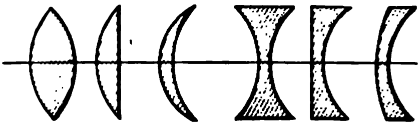


FIG. 426.

To investigate refraction through a lens suppose that Fig. 427 represents a portion of a medium bounded by two spherical surfaces, γ_1, γ_2.

By equation 3 the light-waves converging to the point O have their curvature modified by the first surface, so that it becomes

$$C_1 = C\rho_1 + (1 - \rho_1)\,\gamma_1. \tag{10}$$

This wave surface spreads with uniformly decreasing radius until it reaches the second surface at a distance, t, from the first. Its radius has now become $\frac{1}{C_1} - t$, and its curvature the reciprocal of this, namely,

$$C_1' = \frac{C_1}{1 - C_1 t}. \tag{11}$$

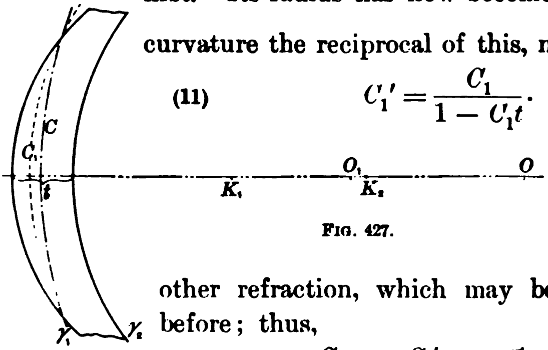


FIG. 427.

At the second surface the wave will undergo another refraction, which may be calculated as before; thus,

$$C_2 = \rho_2 C_1' + \gamma_2\,(1 - \rho_2). \tag{12}$$

In the case most commonly considered, the waves emerge into the air and the thickness is regarded as negligible.

Setting $\rho_2 = \frac{1}{\rho_1} = n$, and $t = 0$, equation 12 becomes

$$C_2 = C + (n-1)(\gamma_1 - \gamma_2), \tag{13}$$

whence it appears that the curvature of a wave is altered by refraction through a thin lens by an amount $(n-1)(\gamma_1 - \gamma_2)$. This expression is called the ***power*** of the lens and may be denoted by ϕ. If the incident waves are plane, $C = 0$ and $C_2 = \phi$, and the radius of the emergent waves is $\frac{1}{\phi}$, which is called the *focal length* of the lens and will be denoted by f.

Equation 13 is often written in the form

$$\frac{1}{r_2} = \frac{1}{r} + \frac{1}{f}, \tag{14}$$

where r_2 is the distance of the image, and r the distance of the image from the lens.

Discussion of Equation. The case of a double convex lens may be used to show the application of the formula and illustrate the relations between the object and image. Thus, making γ_2 negative, C_2 will be positive for C negative and numerically less than ϕ; that is to say, a real image will be formed of all points farther than the focal distance from the lens, and these images will lie between infinity and the principal focus. For C negative and numerically greater than ϕ, C_2 is negative, or the image of any point nearer than the focal distance is virtual, and lies between the lens and infinity. For converging waves, C_2 is always positive; that is, the waves are brought to a focus between the principal focus and the lens.

For double concave lenses equation 13 becomes

$$C_2 = C - (n-1)(\gamma_1 + \gamma_2).$$

The images of all points are virtual, lying between the lens and the principal focus. Converging waves having a curva-

ture numerically greater than ϕ will, after refraction, converge to a point.

557. Magnification by Lenses. — Let K (Fig. 428) be the center of curvature of the surface γ. Let O be a luminous point, O_1 its image, and A another point very close to O. Draw a line through A and K. The image of AO will be at A_1O_1. Let C be the incident wave surface converging to A, and C_1 the refracted wave surface. Then by the law of refraction, since the angles are small, the angle between the surface γ and C_1 is ρ_1 times the angle between γ and C, or

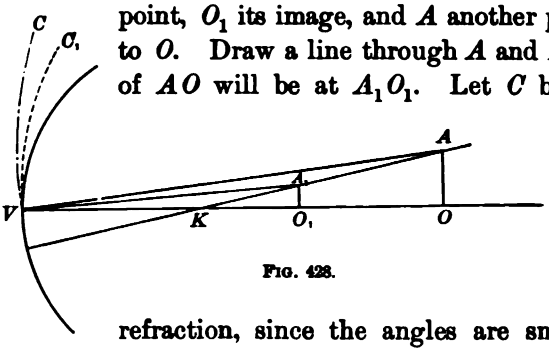


FIG. 428.

$$A_1VO_1 = \rho_1 \cdot AVO,$$

whence, calling the lengths $A_1O_1 = a_1$ and $AO = a$,

$$\frac{a_1}{a} = \rho_1 \frac{C}{C_1}.$$

Similarly, after refraction at the second surface of a thin lens,

$$\frac{a_2}{a_1} = \rho_2 \frac{C_1}{C_2},$$

whence, observing that $\rho_2 \cdot \rho_1 = 1$,

$$\frac{a_2}{a} = \frac{C}{C_2},$$

or the sizes of object and image are directly as their distances from the lens. When the ratio $\frac{a_2}{a}$ is positive, that is, when the object and its image are on the same side of the lens, the image is erect; but when object and image are on opposite sides, the image is inverted.

EXAMPLES.

1. Two plane mirrors are placed parallel and facing each other at a distance of 20 cm. Required the distances of the three nearest images, in each mirror, of an object placed 8 cm. from one of the mirrors.

Ans. $\begin{cases} \text{8, 32, and 48 cm. from first.} \\ \text{12, 28, and 52 cm. from second.} \end{cases}$

2. A circular disc is placed parallel to, and 3 feet in front of, a wall. Required the size and shape of the mirror which, placed on the wall, shall enable an observer standing 8 feet in front of the wall to see the exact outline of the disc.

Ans. A circular mirror, three-fourths of the diameter of the disc.

3. Show that when the sun is shining obliquely on a vertical plane mirror an object placed just in front of the mirror may cast two shadows besides the direct one.

4. AB and AC are two plane mirrors inclined to each other at an angle of 15°. Required the angle at which light-waves should fall on AC from a point in AB, in order that they should, after three reflections, proceed in a direction parallel to AB. *Ans.* 45°.

5. A mirror is made to revolve about a vertical axis 25 times a second. If a horizontal beam of light is allowed to fall on the mirror from a fixed source, required the velocity at which the reflected beam would traverse a circle 78 cm. in diameter having its center on the axis of the mirror. *Ans.* $1.23(10)^4$ cm. / sec.

6. Let AB and CB be two mirrors inclined to each other at an angle, α. Also, let p' be the image in AB of any point, p, placed between the mirrors, and p'' the image of p' in CB. Show that the angular separation of pBp'' is twice the angle between the mirrors.

7. An object 0.96 cm. long is placed at a point 35 cm. in front of a concave mirror having a focal length of 30 cm. Required the size and position of the image. *Ans.* 5.8 cm. long; 210 cm. in front.

8. What is the radius of a spherical mirror which forms an image at a distance of 46.2 cm. in front of the mirror when the object is placed 153 cm. from the vertex? *Ans.* $R = -71.0$ cm.

9. Required the radius of curvature and position of a mirror which would form on a wall a three times magnified image of a gas flame at a distance of 80 cm. from the wall.

Ans. $R = -60$ cm.; distance, 120 cm. from wall.

10. What will be the size of the image of the sun formed by a mirror having a radius of 275 cm., the diameter of the sun being taken as 32 min.? *Ans.* 1.28 cm.

11. An object 3.2 cm. long is placed at a distance of 6 cm. in front of a convex mirror having a focal length of 12 cm. What will be the position and size of the image?

Ans. 4 cm. behind; length, 2.1 cm.

12. If an object be placed at a distance of 25 cm. in front of a concave mirror having a curvature of 0.0167 per cm., what will be the position and size of the image?

Ans. 150 cm. behind the mirror, and magnified six times.

13. What will be the apparent depth of a lake 27.3 feet deep?

Ans. 20.5 feet.

14. If an eye immersed in a fluid whose index of refraction is 1.42, look out through a horizontal surface, what will be the greatest apparent zenith distance of a star? *Ans.* 44° 46′.

15. Find the radius of the circle on the upper surface beyond which light-waves, emitted by a luminous point at the bottom of a layer of liquid 4.2 cm. deep and having an index of refraction of 1.25, will cease to emerge. *Ans.* 5.6 cm.

16. When a layer of liquid 4.65 cm. deep is poured upon a dot on a glass plate, the position of its image, as found by the necessary change in the focus of a microscope, is 1.37 cm. above the plate. What is the index of refraction of the liquid? *Ans.* $n = 1.42$.

17. If the index of refraction from water into another liquid is 1.23, what will be the index of refraction for light passing from air into this liquid? *Ans.* $n = 1.64$.

18. The relative index of refraction for two media is 1.26. If the absolute index for the first is 1.38, what will be the velocity of light in the second medium? *Ans.* $1.72(10)^{10}$ cm. / sec.

19. What would be the minimum deviation produced by a prism whose angle is 1.3°, for which $n = 1.54$? *Ans.* 45.4 min.

20. The minimum deviation produced in monochromatic light by a prism whose angle is 45.05° was 26.67°. What is the index of refraction? *Ans.* $n = 1.530$.

21. Prove that the focal power of a glass lens, $n = 1.5$, when immersed in water is only $\frac{1}{4}$ of its power when immersed in air.

22. The radii of curvature of a thin double convex lens are 46.4 cm., and the index of refraction 1.53. What is its focal length? *Ans.* $f = 43.8$ cm.

23. Required the focal length of a thin lens which forms an image at a distance of 30.3 cm. behind the lens, when the object is placed 91.1 cm. in front. *Ans.* $f = 22.7$ cm.

24. An object is placed 59 cm. in front of a positive lens whose focal length is 14.9 cm. Required the magnification. *Ans.* Three times.

25. If the nearest distance of distinct vision for a far-sighted person is 2 ft. 11 in., what should be the focal length of the spectacles he would require for reading? *Ans.* $f = 14$ in.

26. If the greatest distance of distinct vision for a myopic eye is 3.9 in., what should be the focal length of the proper reading spectacles? *Ans.* $f = 6.4$ in.

27. A positive lens placed at a distance of 5.2 cm. from a luminous object forms an image on a screen. When the lens is moved a distance of 23 cm. nearer the screen, another image is formed. What is the focal length of the lens? *Ans.* $f = 4.4$ cm.

28. A positive lens placed at a distance of 12.7 cm. from a screen forms on it an image six times the size of the object. What is the focal length of the lens? *Ans.* $f = 10.9$ cm.

29. A luminous object placed a distance, d, in front of a screen has an image thrown on the latter by means of a convex lens. On moving the lens toward the object another image is formed which is a times as great as the first. Required the focal length of the lens.

$$\textit{Ans.}\quad f = \frac{\sqrt{a}}{(1 + \sqrt{a})^2}\, d.$$

30. A luminous point is placed on the axis of a hemispherical lens of radius R, at a distance, d, in front of the plane surface. If the image formed by reflection at this surface coincides with that formed by refraction at the plane and reflection at the spherical surface, show that the index of refraction of the glass is $n = \dfrac{R}{R - 2d}$.

31. Show that if light-waves fall at an angle of 60° on a sphere whose refractive index is $\sqrt{3}$, they will emerge after one internal reflection parallel to their original direction.

32. A small air bubble is embedded in a sphere of glass whose radius is 7.03 cm. and index of refraction 1.4, at a distance of 5.98 cm. from the nearest point of the surface. What will be its apparent depth when observed from this side? *Ans.* 5.64 cm.

33. The upper surface of a thin convex lens has a radius of curvature of 10 in., and an index of refraction of 1.6. The lower face with a radius of 15 in. is just immersed in water. Find the position of the two principal foci.

Ans. Downward waves, $f_1 = 17.1$ in.
Upward waves, $f_2 = 12.9$ in.

34. If the eye be placed close to the surface of a sphere of glass, show that the image which the eye sees of itself would be $\frac{3}{5}$ of its natural size.

35. Find the focal length of a glass sphere.

Ans. $f = \dfrac{1 - \frac{n}{2}}{n - 1} R.$

CHAPTER XXXVIII.

ELEMENTARY THEORY OF OPTICAL INSTRUMENTS.

558. Camera Obscura. — If a white screen be placed at the position A_1B_1 (Fig. 429), where the real image of any object, AB, is formed by a lens, L, and surrounded by the blackened walls of a box so that no light-waves except those from the object AB can fall upon it, each point of the paper will appear bright or dark according to the brightness of the corresponding point in the object; in short, the surface of the paper will appear as a faithful but inverted picture of the object. An instrument so constructed is called a *camera obscura.* Used formerly only as a scientific toy, or occasionally as an aid in drawing the outlines of an object, it has become, since the invention of the various methods of fixing an image on a sensitive plate, one of the most important optical instruments. Since it is necessary to have the surface upon which the picture is formed at a distance from the lens depending upon the distance to the object, the sides of the photographic camera are made extensible like the flexible portion of a bellows, and the adjustment of the focus is made before the sensitive plate is introduced. The magic lantern and the solar microscope are cameras in which the object is quite close to the lens and its image remote. The only additional feature in each of these instruments is an arrangement by which the object may be very strongly illuminated.

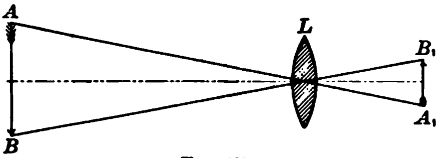


FIG. 429.

559. The Eye. — Considered as an optical instrument, the eye is simply a camera obscura. The lens system consists of the *cornea*, *a* (Fig. 430), a chamber, *m*, filled with the aqueous humor, and a harder transparent body, *o*, known as the *crystalline lens*. The chamber of the eye is darkened by

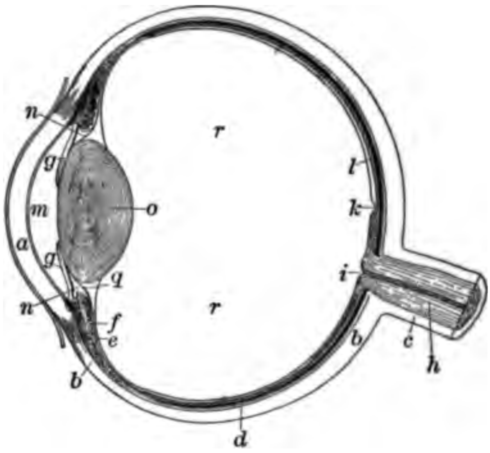


FIG. 430.

a, cornea; *b*, sclerotic; *c*, sheath of optic nerve; *d*, choroid; *e*, ciliary muscle; *f*, ciliary process; *gg*, iris; *h*, optic nerve with artery in center; *i*, papilla; *k*, fovea centralis; *l*, retina; *m*, anterior chamber of aqueous humor; *n*, posterior chamber of aqueous humor; *o*, crystalline lens; *q*, suspensory ligament of lens; *r*, vitreous humor.

a black opaque membrane (*choroid*), *d*, just within the white skin (*sclerotic*), *b*, which forms the visible outer substance of the eyeball. Inside this dark coat is the delicate white membrane of nerve ends, *l*, called the *retina*, and forming the sensitive surface upon which the images of external objects are depicted. The cavity of the eyeball, *r*, is filled with a transparent gelatinous mass, known as the *vitreous humor*. Just in front of the crystalline lens is a colored

opaque diaphragm (*iris*), *gg*, pierced with a circular aperture (*pupil*), by which the quantity of light admitted to the eye is regulated. The only striking difference, from an optical standpoint, between the eye and the photographic camera is the mode of adapting the apparatus to the varying distance of the object. Instead of modifying the distance between the screen and the lens, the power of the lens is altered by changing its thickness at the middle. This capacity for "accommodating" the power of the lens to the use required is very remarkable in young children, enabling them to see objects with perfect sharpness from a distance of three or four inches up to infinity. A great diminution in the accommodation occurs about the twentieth and another about the fortieth year. If the eye during the earlier age is employed almost exclusively for near objects, the lens is apt to assume a permanently thickened form, so that it is too powerful for waves having a very remote source. Such an eye is called short-sighted, or myopic. In order to see distant objects, its power must be reduced by means of negative lenses, *i.e.* spectacle glasses thinner in the middle than at the edge.

For purposes of comparison, the distance of distinct vision for an object or its image in front of the eye is arbitrarily chosen as ten inches. Thus, suppose, for instance, that the greatest distance of distinct vision in a myopic person has been found by trial to be r inches. It is then obvious that the focal power, $\phi = \frac{1}{f}$, of the negative lens which shall reduce the original curvature, $\frac{1}{r}$, of the waves to $\frac{1}{10\text{ in.}}$ will be given by the relation

$$(1) \qquad \frac{1}{r} - \phi = \frac{1}{10\text{ in.}},$$

from which the focal length may be calculated at once.

In eyes having no disposition toward myopia, the power of thickening the lens at will is gradually lost, and the form becomes more and more continuously that proper for vision of remote objects. Such an eye is called far-sighted, or presbyopic. For the purpose of clear vision of near objects its power must be increased by the aid of positive lenses. If the nearest distance of distinct vision in a presbyopic eye has been found to be r inches, then the focal power of the lens which will be necessary to increase the curvature of the wave to $\frac{1}{10\text{ in.}}$ will be given by

$$\frac{1}{r} + \phi = \frac{1}{10\text{ in.}} \qquad (2)$$

In the normal eye the loss of range of accommodation is probably continuous through life, but it does not usually progress far enough to take the nearest point of distinct vision beyond convenient reach before the fifth decade, when the necessity of aids to distinct vision becomes evident. Many persons, however, are born with eyes myopic, either from excessive power of the lens or cornea, or from abnormal axial length of the eyeball; so, too, instances are not rare where deficiency of refractive power is congenital, in which cases positive lenses must be used even for remote objects. Another defect by no means rare is for the eye to have different powers in different planes. It can be recognized by the differing distinctness with which horizontal and vertical lines in a brick wall may be seen, or by the elongated appearance of a star. This fault is known as *astigmatism*. It may be compensated by a lens with cylindrical instead of spherical surfaces.

Since the eye is optically a camera obscura, it follows that the image depicted on the retina is inverted with respect to the object. Why this inversion does not appear in the sen-

sation is a question of psychology rather than of optics. It is sufficient to note here that an inverted image on the retina corresponds to the sensation of an erect object and *vice versa.*

560. Simple Microscope. — When an object or its image is within the range of distinct vision, *i.e.* in front of the eye and at a greater distance than six inches, it can be distinctly seen if large and bright. If an object is too small to be seen with distinctness, it may be made to appear larger by bringing it nearer the eye, so that at half the distance it appears twice as large. But if this means of increasing the apparent size is carried too far, the power of the eye becomes insufficient to change convex wave surfaces from the object to concave ones having their centers on the retina. Under these circumstances the power of the eye lens may be increased by means of a positive lens close to the eye. A lens so used (Fig. 431) is called a simple microscope.

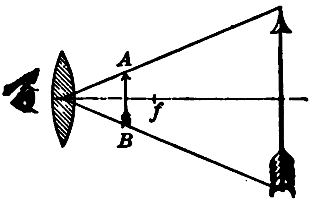


Fig. 431.

The magnifying power of a microscope is defined as the ratio of the apparent size of the image to the size of the object as seen at a distance of ten inches. The object *AB* (Fig. 431) is always a little nearer the lens than the focus; but if the eye be placed very near the lens, ten inches divided by the focal length may be taken as an approximate value of the magnifying power of a simple microscope.

561. Galilean Telescope. — The principle of the telescope seems to have been first discovered by Franz Lippershey, a spectacle maker of Middelburg, in 1608. In the following year Galileo, while visiting Venice, learned for the first time that a marvelous instrument had been discovered in Holland,

which would enable an observer to see a distant object with the same distinctness as if it were at only a small fraction of its real distance. Soon after his return to Padua, where he held the position of professor of mathematics, he attacked the problem independently, and with such success that in a few months he sent to the Venetian Senate a more perfect instrument than they had been able to procure from Holland. Six months later, by means of a telescope magnifying but thirty times, he discovered four satellites of Jupiter, soon after following it by the discovery of the mountains on the moon and the variable phases of Venus, which incontestably established the Copernican theory of the solar system.

On account of his discoveries and improvements, the name of Galileo has always been connected with the form of telescope which he used.

The Galilean telescope consists of a negative lens, *a* (Fig. 432), close to the eye, and a positive lens, *b*, at a certain distance from the first depending on its power. For the sake of simplicity it may be assumed that the negative lens of the telescope just neutralizes the lens of the eye. This supposition entails no loss of generality in the conclusion drawn from it, and is in substantial accord with the fact in the instrument as it now survives. In this case the waves suffer no change of curvature in passing into the eye, and if the lens have such a position that waves from a distant source have their centers of curvature after passing it on the retina, the conditions of distinct vision are met. The effect of the

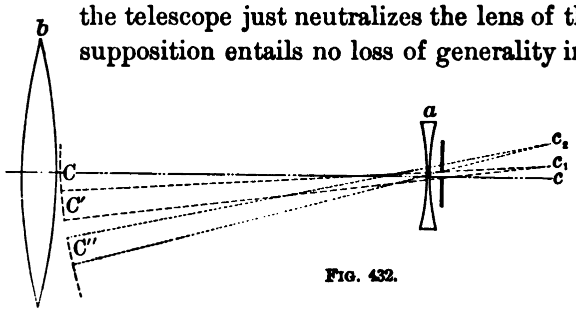


FIG. 432.

instrument is thus merely to increase the virtual size of the eye, and with it, as appears from the theory of the camera, the size of the image on the retina in the same ratio. In one particular only does this apparatus, the Galilean telescope and the eye, differ from a large eye, and in that difference lies the limitation of this form of instrument. In the eye the iris, which limits the portion of the lens upon which the light-waves fall, is close to the lens and remote from the retina. In this magnified eye the iris is relatively near the retina and remote from the lens. From this it follows that light corresponding to different points of the image passes through different parts of the lens b. This is evident from Fig. 432, for waves which pass through the pupil of the eye and converge to c_1 come from a portion of the objective near C', while those waves which produce the point c in the retinal image proceed from a different portion, C. A point in the object which gives rise to the waves which, after passing the objective, have their center at c_2, cannot be seen at all, since they are stopped by the iris. Thus, the extent of that portion of the object which can be seen with such a telescope depends on the diameter of the objective and that of the pupil of the eye. If the magnifying power is considerable, this extent is very small with moderate-sized lenses. Very large lenses are not only expensive but impracticable for short telescopes, because the wave surfaces transmitted by them are no longer spherical.

An instrument having this defect is said to have a small field. For this reason this type of telescope is only used where very moderate magnification, say from two to six times, is required, and where extent of field is sacrificed to compactness and lightness. In opera-glasses, with a magnification rarely over three, it serves well, as it does also for use on the sextant.

562. Astronomical Telescope.— The form of telescope ordinarily called the astronomical telescope may be readily understood from a consideration of Fig. 429.

There the objective forms an image of distant objects on the screen. Now, imagine the eye just at the center of the lens. If turned toward the object, this would appear of a certain magnitude; if turned toward the screen, the inverted image would appear of the same magnitude as the object itself, since the magnitude of the image and the object are directly

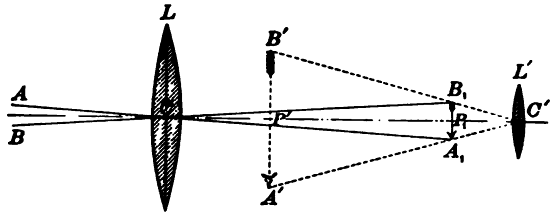


FIG. 433.

as their distances from the lens. If, now, the eye turned toward the screen be approached toward it, the image will appear larger than the object in the ratio of the respective distances of the lens and the eye from the image. Hence, if the focal length of the objective be denoted by F, the magnification will be $\frac{F}{10 \text{ in.}}$.

This illustrates the general principle involved, though in practice the screen is omitted and the eye placed in the prolongation of the axis of the lens. The object is thus seen inverted, but this is of no consequence in astronomical observations. The camera and screen are still usefully employed, however, in certain observations of the sun. It was in this way that the first observed transit of Venus across the face of the sun was seen by Horrocks in 1639.

If the eye is brought very near the image, in order to

secure greater magnification, its power must be increased by a lens used as a simple microscope (Fig. 433). Such a lens, or, preferably, system of lenses, is called the *eye-piece*, or *ocular*. The magnification then becomes the focal length of the objective divided by the focal length of the ocular, or $\frac{F}{f}$.

Since the ocular will form an image of everything in front of it and sufficiently removed, it will form an image of the objective. If the telescope be directed toward a bright sky, an image of the objective, known as the ocular circle, will be formed very near the eye lens. Now, since the objective and its image are proportional to their respective distances from the ocular, *i.e.* to the focal lengths of the objective and the ocular, the magnification of the telescope is the ratio of the diameter of the objective to that of the ocular circle. In practice these values are more easily measured than the focal lengths.

The ocular circle is the smallest area through which light passes after leaving the telescope, and hence marks the best point for placing the eye. If the pupil is smaller than the ocular circle, it is obvious that not all the light transmitted through the objective will reach the retina. The lowest power, then, that can be used with a telescope, and utilize the full capacity of the instrument, is that which makes the ocular circle just the diameter of the pupil. This diameter is variable, but in a feeble light it may be taken as from a fifth to a sixth of an inch. It follows from the ratio given above that the lowest power which can be used to full advantage on any telescope is five or six times the diameter of the objective in inches. If, on the other hand, the magnification is increased to some six times this amount, the vision begins to be impaired for reasons which will be more fully discussed in Art. 584.

563. Reflecting Telescopes. — The early opticians found that when they attempted to increase the diameter of a telescope they were obliged to increase its length in a greater ratio in order to secure distinct vision.

The cause of the indistinctness of small telescopes was explained by Newton, after his discovery of dispersion, as due to the fact that light of different colors coming from a point was differently refracted, so that if the ocular was placed for one particular color it would not be in the right position for any of the others, whence the image of a star or

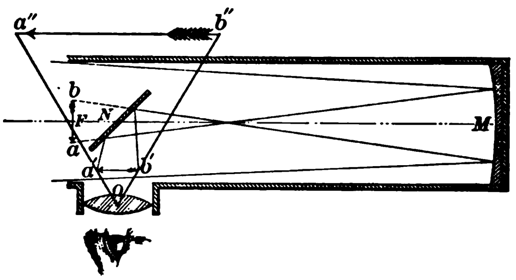


FIG. 434.

planet appeared surrounded by a fringe of colored light. Having satisfied himself that this phenomenon of chromatic aberration was an insuperable obstacle to the further improvement of the refracting telescope, he turned his attention to another form, which had been suggested a number of years earlier, in which an image was formed by reflection from a concave mirror. The arrangement adopted by Newton is shown in Fig. 434. M is the mirror of speculum metal, which would form an image of a distant object at ab; N is a small plane mirror which changes the position of the real image from ab to $a'b'$, and O the ocular producing a magnified virtual image of $a'b'$ at $a''b''$.

The ocular is placed on the side of the tube in order that the head of the observer may not obstruct the light.

The reflecting telescope was gradually improved in the hands of various makers, notably by Sir William Herschel, and during the eighteenth century entirely supplanted the refracting type. However, since the discovery of the possibility of correcting chromatic aberration, the refracting telescope has, in turn, nearly displaced the reflecting type.

564. Compound Microscope. — The compound microscope may be conveniently regarded as an inverting telescope adjusted for an object very near the objective. The geo-

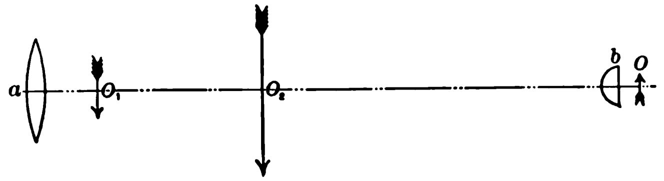


Fig. 435.

metrical principles involved are illustrated in Fig. 435, where b is the objective which forms an image of the object O at O_1, and a is an eye-piece through which O_1 is observed.

The magnification by the objective is evidently the ratio of the distance $b\,O_1$ to $b\,O$, or, calling the length of the tube L, and the focal length of the objective F, approximately $\frac{L}{F}$. Accordingly, since the magnification of the ocular is $\frac{10^{in}}{f}$, the total magnification is $\frac{10^{in}\,L}{f\,F}$.

The compound microscope is in no respect superior in theory to the simple microscope, but in practice it is difficult to make simple microscopes of very great power, say of a

magnification much greater than 250, because of the extreme minuteness of the lenses required.

565. Terrestrial Telescope. — If the ocular of the astronomical telescope be replaced by a compound microscope, an erect image of the distant object will be seen. This arrangement constitutes the terrestrial telescope, or spy-glass. It is necessarily longer than the astronomical telescope, and less perfect because it is subject to the unavoidable defects of a larger number of lenses.

An image may be inverted by successive reflection as well as by refraction, and, with a proper sequence of reflections, as was first shown by Poro, it is possible to invert an image without changing the direction of the light from it. Such devices are coming into extensive use for terrestrial telescopes of low power largely on account of their compactness.

In the discussion of the purely geometrical principles involved in the more important forms of optical instruments, the magnification has been shown to depend upon the powers of the lenses in a simple manner. It will be shown in Chap. XLI that the efficiency of the instruments is determined almost wholly by the effective diameters of the lenses, and that their powers play but an insignificant rôle.

566. Sextant. — The sextant (Fig. 436) is an instrument, invented by Hadley, for the purpose of measuring the angular displacement between two distant objects. *FF* is an arm pivoted at the center of a sixty degree sector, and traversing a graduated arc, *AA*. Fastened to this arm and turning with it is a mirror, *B*. A second mirror, *C*, silvered only on the half next to the frame, is fixed in a position parallel to the first mirror, when the radial arm is at zero. *DE* is a small telescope directed toward *C*, and intended to assist the observer in viewing a distant object.

Suppose that light is falling on the instrument from two distant points in the directions HC and SB respectively. Then, by turning the arm F through a proper angle, it is possible to arrange the mirrors so that one-half of the pupil placed at E shall receive light coming from H through the unsilvered portion of C, while the other half receives light from the second source S after reflection from the surfaces of both mirrors. It may now be shown that when

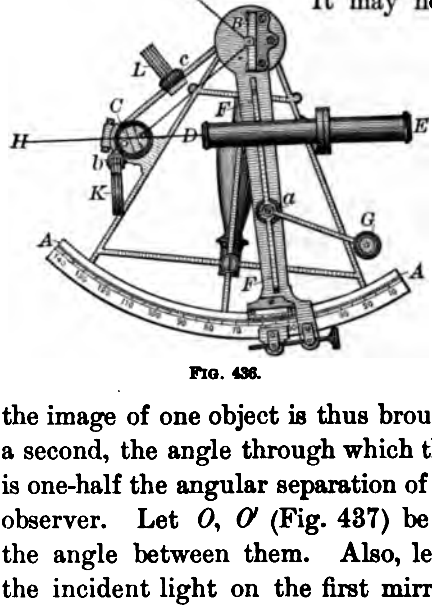


FIG. 436.

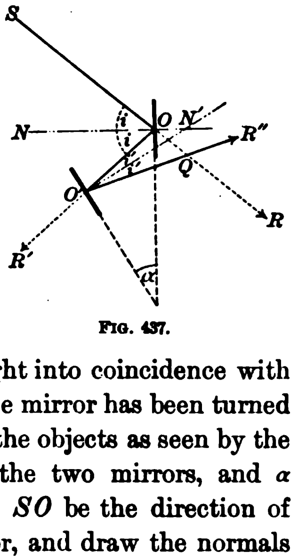


FIG. 437.

the image of one object is thus brought into coincidence with a second, the angle through which the mirror has been turned is one-half the angular separation of the objects as seen by the observer. Let O, O' (Fig. 437) be the two mirrors, and α the angle between them. Also, let SO be the direction of the incident light on the first mirror, and draw the normals NO, $N'O'$. Measuring the angles in the usual direction, let

$\delta = ROR'$, the deviation at the first mirror,

$\delta' = R'O'R''$, the deviation at the second mirror,

$i = SON$, the angle of incidence on the first mirror,

$i' = OO'N'$, the angle of incidence on the second mirror.

Then, by the law of reflection,

$$\delta = \pi + 2i, \qquad \delta' = \pi - 2i.$$

But in the triangle $OO'N'$ the exterior angle NOO' is equal to the sum of the opposite interior angles $ON'O'$ and $OO'N'$; that is,

$$i = i' + \alpha,$$

or

$$i - i' = \alpha;$$

therefore the deflection is

$$\delta + \delta' = 2\pi + 2(i - i') = 2\pi + 2\alpha,$$

or the angle $RQR'' = 2\alpha$.

For convenience of observation the limb of the sector is divided so as to read half-degrees as if they were whole degrees. Several plates of colored glass are provided at L and K in order to diminish the intensity of the light when it is too bright for the eye.

The sextant is especially useful at sea in determining the distance of the sun above the horizon at any time, since the apparent coincidence of these objects is not affected by the motion of the ship. If the altitude of the sun be observed when it is on the meridian, the latitude of the place may be found when the solar declination is known.

Likewise, if the altitude of the sun be taken in the middle of the afternoon when it is near the prime vertical, the local time of the place may be found in terms of the known latitude and the time at Greenwich, as shown by the ship's chronometer, and hence the longitude may be calculated.

EXAMPLES.

1. What will be the magnifying power of a telescope of which the objective has a focal length of 610 cm. and the ocular one of 1.27 cm.? *Ans.* 480.

2. What would be the magnifying power of an opera-glass if the focal length of the objective be 10.2 cm. and that of the ocular 3.5 cm.? *Ans.* 2.9.

CHAPTER XXXIX.

INTERFERENCE.

567. Phenomena of Limited Wave Surfaces. — Suppose Fig. 438 to represent a wave-front limited by the parallel edges *ab* of a slit in a screen and moving toward the center *p*, which is thus by definition the image of the point from which the wave took its origin. According to Huyghens's Principle, each point of the wave-front *ab* must be regarded as the center of a disturbance which is propagated in all directions through the medium. Let p_1 be a point so chosen that its distance from *a* is a half wave-length greater than its distance from *b*; that is, so that *aa′* equals a half wave-length. Then a disturbance setting out

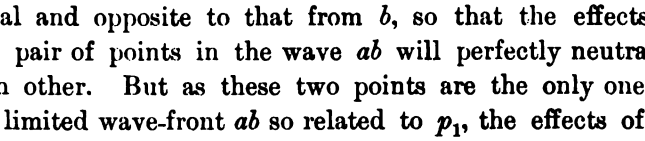


FIG. 438.

from *a* will reach p_1 at the same time as one from *b* starting a half period later. In short, the displacement at p_1 due to the wavelet from *a* will be exactly equal and opposite to that from *b*, so that the effects of this pair of points in the wave *ab* will perfectly neutralize each other. But as these two points are the only ones in the limited wave-front *ab* so related to p_1, the effects of all the other pairs of points which might be chosen in the wave-front, symmetrically placed with respect to its middle point, will only partially counteract each other. For points between *p* and p_1 the destruction of the elementary waves is

even less complete, for there is no pair of points which wholly destroy each other's effect. Next consider the point p_2 such that its distance from a is a whole wave-length greater than its distance from b. In this case the disturbance from the middle point of the wave-front will be a half wave-length behind that from b and will destroy its effect; but for every point between b and the middle of the wave-front a corresponding point between the middle and a can be found, which is a half wave-length farther from p_2; hence the effects of all the elementary waves at p_2 will be *nil*, and the medium there will remain undisturbed. By extension of this reasoning it appears that in general there will be motion due to the effect of bounded wave-front except where the difference of distance from a and b is a whole number of wave-lengths. When the difference is an odd number of half wave-lengths, the disturbance is greater than at any closely lying point, because the conditions of self-destruction are most widely departed from. On the other hand, it is obvious that the absolute value of the disturbance decreases very rapidly on leaving the position of the geometrical image p, because a larger and larger number of pairs of mutually destructive centers can be found. The light which is found outside the geometrical image is called *diffracted light*, and a large class of analogous phenomena are embraced under the general term *diffraction*.

From the preceding discussion the conclusion may be drawn that a limited concave wave-front forms not a simple image at its geometrical center, but a series of images of which the middle one corresponds in place with the geometrical image and is by far the strongest, while it is symmetrically flanked on both sides by a series of secondary images rapidly diminishing in brightness. It is further evident from the figure that pp_2, which is one-half the width of the central

image, bears the same ratio to the distance ap that aa' does to ab; that is,

$$(1) \qquad pp_2 = ap\frac{\lambda}{ab}.$$

568. Diffraction through a Circular Aperture. — The case in which the wave-front is bounded by a circular aperture is the most common and interesting case in optics, for it is that of practically all optical instruments. The phenomena are rather more complex, but not different in kind. As might be expected, the diffracted image in this case becomes a circular area surrounded by concentric rings, but their radii are slightly different from the values derived for the fringes in Fig. 438. Thus, calculation shows that the radius of the first dark ring is 1.2 times as great as from p to p_2.

569. Conditions necessary for Observance of Diffraction Phenomena. — The conditions implied in the preceding reasoning and necessary for the realization of the results described are somewhat rigorous. First, in order to secure regular phenomena about the region p, the series of waves must have a uniform length, since the distance from the primary to the secondary images depends on this length. Again, if the opening through which the waves come is less than a wave-length, there is no point, such as p_2, whose distance from one edge of the aperture is a whole wave-length greater than that from the other; consequently the most striking peculiarity of the effect, *i.e.* regions of quiescence, are wholly wanting. In the case of sound-waves, in which the length for a note of medium pitch is about four feet, it is obvious that the aperture would have to be a number of feet across in order to produce the required effect. But in cases where the apertures are as great as this, being of the same order as the dimensions of the enclosures in which the observations would be likely

to be made, the reflection from the surrounding wall would entirely mask the effects sought; at least to one not guided by theory in his research. Thus, it is not surprising that this particular class of phenomena in sound-waves failed of recognition until after their discovery in the case of light-waves.

Finally, if the aperture is many times larger than the length of the wave, the brighter secondary images will lie very close to the primary, so close in fact as to escape our powers of perception. This, indeed, is generally the case with light-waves.

Since to the eye a star appears as a round point of light and not surrounded by a series of concentric circles, although the waves from the source are bounded by the circular edge of the iris, the conclusion may be drawn that light-waves are very many times shorter than the diameter of the pupil, or, in other words, an eighth of an inch is very many times greater than the wave-length of light.

By looking through a needle hole in a card at a very bright point, such as a distant electric light, or more conveniently a bright bead or a thermometer bulb in the sunshine, the central disc and the concentric rings become evident at once. Moreover, in accordance with theory, the smaller the hole in the card, the larger the disc and its surrounding rings, though of course the fainter, because less light is allowed to enter the eye. If the luminous point, or the artificial star, as it is sometimes called, is very brilliant, two or three or even more rings may be observed; but if a less brilliant source is employed, the outer rings, which decrease very rapidly in brightness, will be imperceptibly faint.

570. Diffraction through a Double Aperture. — The modification in the image produced by a concave wave surface passing through two small holes in a screen is of special

interest because leading to a ready means of measuring the wave-length of light.

Suppose *ab* and *a'b'* (Fig. 439) to represent two circular holes through which the wave-front whose center is at *p* passes. Then the light which passes through *ab* forms an image in the region about *p* consisting of a central disc and concentric rings, as has been already shown. So, too, the light which passes through *a'b'* forms a similar image at the same place. If the waves which pass through the former opening have no definite relations to those which pass through the latter, as would be the case, for example, if the waves through *ab* came from one artificial star and those through *a'b'* from another, the effect will be to make the image twice as bright, since twice as much light passes through the screen. But if the waves at the two apertures are

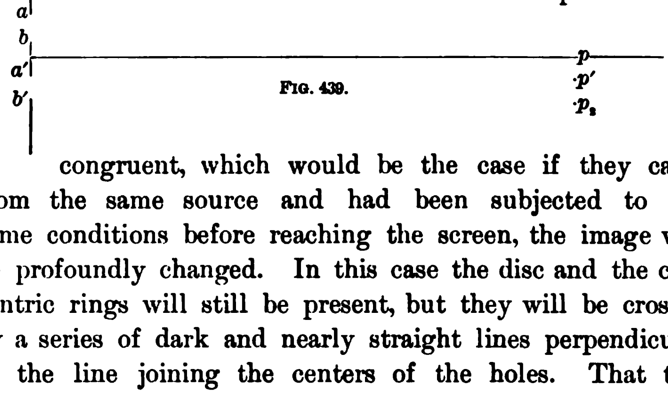


FIG. 439.

congruent, which would be the case if they came from the same source and had been subjected to the same conditions before reaching the screen, the image will be profoundly changed. In this case the disc and the concentric rings will still be present, but they will be crossed by a series of dark and nearly straight lines perpendicular to the line joining the centers of the holes. That this should be so follows from the consideration of the principles involved. The distance from the center to the first dark ring, which may be called the radius of the primary image, is, as has been explained, 1.2 times the wave-length multiplied by the ratio of *ap* to *ab*. But, starting from the point *p* long before the point p_2, the dark region due to *ab* alone, is reached, we come to a position, p', such that the dif-

ference of its distances from a and a' is a half wave-length, and hence the effect produced at p' by the portion of the wave from a is destroyed by that produced by a'. If, however, the center a is neutralized by the center a', the effect of every other point within the region ab will be neutralized by the corresponding point in $a'b'$. Consequently there will be no light at p'. But this reasoning is equally applicable to a point, p'', which is at three half wave-lengths' greater distance from a than from a', and also when this difference of path is any odd number of half wave-lengths. Hence there will be a series of dark lines as described.

It will be observed that the diameter of the image is inversely as the diameter of the aperture, but that the distance apart of the dark lines is inversely as the distance between the holes.

This phenomenon may be easily seen by making two needle holes in a card at a distance considerably less than the diameter of the pupil of the eye, and looking through them at an artificial star.

571. Measurement of Wave-Length. — The preceding experiment affords a method of determining the wave-length of light with considerable precision by means of a very simple apparatus. For this purpose make a series of pairs of holes in a card with a needle and look through them at an artificial star. If the pair of holes are separated by an interval of a twelfth of an inch or more, no lines across the image of the star will be seen; but if the interval is a twentieth of an inch or less, the lines become very distinct. Having selected a pair of holes which are at the limit of the resolving power of the eye, *i.e.* at such distance apart that the lines can just be seen, let the distance between the holes be measured with a finely divided scale. If this scale is divided to hundredths

of an inch, it will be possible to measure this distance, which may be denoted by D, much closer than a two-hundredth of an inch. Neither of the other quantities, $p'p''$, ap, can be easily measured directly, but their ratio may be found as follows: Draw a series of parallel lines on a piece of white paper in ink, making the width of the lines about equal to the spaces. Fasten this paper to the wall and find the distance at which they can just be seen as separate lines when brightly illuminated. The ratio of the distance apart of the lines to this distance is the required ratio, since by supposition the system of interference lines was also just visible as separate lines. It then follows, from the discussion already given in connection with Fig. 438, that the wave-length of light is D times this ratio. For instance, it was found by one observer that, looking through certain pairs of holes at a thermometer bulb in sunshine, no lines could be seen through the first pair of holes, very distinct lines were visible through a second, and the finest possible lines through a third. Measurement of the intervals gave 0.08 in., 0.05 in., and 0.065 in., respectively. For that observer's eye, then, $D = 0.065$ in. A series of parallel lines $\frac{1}{20}$ inch apart were next drawn on a piece of paper, and it was found that they could just be seen as separate lines, in full sunlight, at a distance of twelve feet. The ratio of the interval between the lines to their distance from the observer was accordingly $\frac{1}{2880}$, whence the wave-length of light is $\frac{0.065}{2880}$ in., or $\frac{1}{44000}$ of an inch.

572. Fresnel's Experiments.—Instead of allowing the light-waves to pass through two holes, as in Fig. 439, an experiment due to Young, certain other arrangements of apparatus proposed by Fresnel are frequently used, when all the measurements on the interference fringes may be made directly.

In one of these a piece of glass, known as a ***biprism***, having an angle, ABC (Fig. 440), differing very little from 180°, is placed before a brightly illuminated slit, s, of which it forms by refraction two images, s' and s'', near together. The

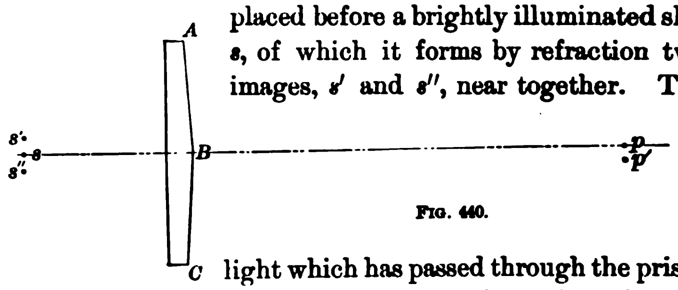


Fig. 440.

light which has passed through the prism very near B is in condition to interfere and will form fringes near p', as if s', s'' were real sources of light.

In another arrangement Fresnel substituted for the prism two mirrors, BA, BC (Fig. 441), whose planes make a very small angle with each other, so as to form two images

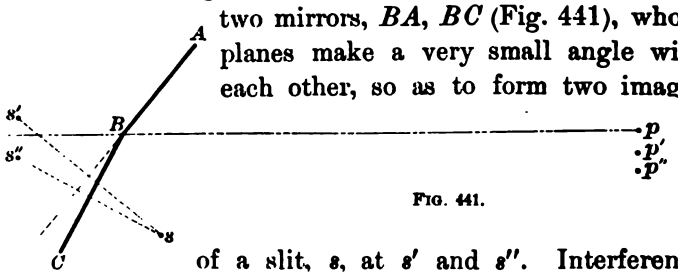


Fig. 441.

of a slit, s, at s' and s''. Interference fringes will be found at p', p'', where the difference of path is an even multiple of a wave-length.

573. Interference Phenomena in White Light. — The explanation of interference phenomena may be quite easily extended so as to include the color effects due to varying wave-lengths. It is only necessary to observe that the images produced have dimensions in direct ratio to the wave-length of the light which forms it. Accordingly the image of a white artificial star consists of a disc of which the center is white, since all the waves are there represented, but having a red or orange margin. The rings immediately surrounding

the disc are blue on the inner side and red on the outer. Though this is the true description of the image, it is impossible to recognize it as such because of the limit of our perceptions; for if the aperture be made large so as to make the colors brilliant, the disc and rings will be too small to be clearly perceived. On the other hand, if the opening be very small, so that the various features of the image are large enough to be obvious, the light will be so faint that the colors are unrecognizable, just as we are unable to name colors readily in even very bright moonlight. With two or more apertures, however, better success may be obtained. In the case of two holes the resulting image may be regarded as composed of an indefinite number, all having the same center, but of sizes increasing regularly from the violet to the red. As we go outward from the center the chromatic separation will become greater, until finally we reach a point beyond which every color, though of course not every wave-length, will be represented at all points. In all such regions the image will appear white and the immediate effects of interference vanish. This limit is found to be practically reached in the case of white light in the eighth or ninth band, so that in white light no more than seventeen or nineteen interference bands are visible even under favorable circumstances, though with light of one color it is sometimes possible to see many thousands.

The chief difficulty in seeing the colors in the experiment with two apertures is the fineness of the lines and the faintness of the colors; but with holes a thirtieth of an inch apart or less, the lines are sufficiently coarse, though if many bands are to be seen the holes should be small.

Since greatly increasing the number of apertures, provided they are symmetrically arranged, produces little change except increased brightness, the color effects may be made

very obvious by this means. An artificial star seen through the web of a uniform feather will thus show very beautiful effects.

Though nature and art present numerous examples of such structure, this class of phenomena is not familiarly known, because these bodies are more often looked at than looked through. Perhaps the only common example where both regularity of structure of the screen and smallness of the source of light are met is when an electric light is observed through a silk umbrella.

574. Gratings. — A diffraction grating is a piece of transparent glass or polished metal ruled with a great number of equidistant parallel lines by means of a very fine diamond point. These grooves are practically opaque, for they scatter the light in all directions. If light passing through a slit be allowed to fall on such a grating held with the rulings parallel to the slit, a number of spectra, of great regularity and purity where they do not overlap, may be observed on either side of the direct path of the light.

Gratings are of much practical interest as furnishing a perfectly normal spectrum and the most accurate means for measuring the wave-lengths of light.

To show how this may be done, suppose that Fig. 442 represents a highly magnified portion of a grating, and that plane-waves are falling on the grating from the left. Each point of those portions of the wave-fronts which fall in the spaces becomes a new center of disturbance and propagates light in all directions. Let p be a distant point in the direction ap such that the difference in path in the wave which reaches it from a is a whole wave-length behind that from c. Then, dividing up the spaces ab and cd into pairs of points related in the same way as a and c, it follows that the effect

of the limited wave-fronts *ab* and *cd* is to produce light at *p*. Similarly, the light sent by each consecutive pair of wave-fronts in the direction *ap* will be in the same phase as that from the first pair and simply add to the intensity at *p*. Call the distance between the rulings *ac*, *s*, and the angle *bap*, θ, then from the figure

$$\text{(2)} \qquad \sin\theta = \frac{ac'}{ac} = \frac{\lambda}{s}.$$

Now, as the angle θ may easily be found by observation, and *s* may be measured, λ may be calculated very simply. If the incident light be white, as the angle θ increases with

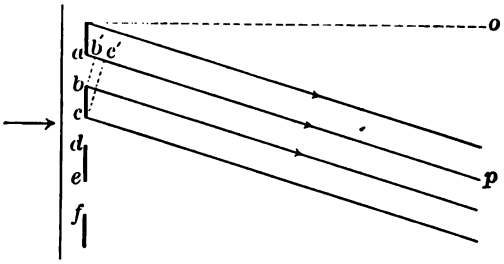


Fig. 442.

the wave-length the resulting spectrum will be arranged with the violet deviated least, and the red most.

If the direction *ap* be chosen so that *ac'* is equal to 2λ, 3λ, etc., similar spectra of the 2d, 3d, or higher orders are formed on each side of the central image, though the purity of those beyond the second may be greatly impaired by the overlapping of the lower spectra. In actual observations the flatness of the incident waves is secured by allowing them to pass through a lens system, called a collimator, and the measurement of the angle θ is assisted by the use of a telescope with cross wires.

575. Reflection Gratings. — The gratings just described are known as transmission gratings, but entirely similar results may be obtained by using a piece of polished speculum metal ruled in the same way. The only modification in the theory is the replacing of the slit by its image produced by the polished surface. The mechanical difficulties of ruling upon glass are somewhat greater than upon metal.

Professor Rowland has introduced the important modification of ruling the gratings on concave spherical surfaces, and is thus enabled to dispense with the collimator and the object glass of the observing telescope. Such a grating is especially adapted for photographic work. By the use of the grating at Johns Hopkins University having 160,000 lines in the space of six inches, the solar spectrum has been photographed in sections which, placed end to end, extend over a length of thirty feet, and measurements of the wave-lengths of light made which are universally recognized as the best now attainable.

576. Wave-Lengths. — The wave-lengths in air of the principal lines of the solar spectrum are shown in the following table. These measures are expressed in what is known as the Ångström unit, or tenth-meter, *i.e.* $\frac{1}{10^{10}}$ meter, or 0.00000001 cm.

$A = 7621$	$b_1 = 5183.791$
$B = 6870.186$	$F = 4861.527$
$C = 6563.054$	$G = 4340.634$
$D_1 = 5896.357$	$h = 4102.000$
$D_2 = 5890.186$	$H = 3968.625$
$E_1 = 5270.495$	$K = 3933.825$
$E_2 = 5269.723$	

Taking the velocity of light as $3(10)^{10}$ cm. per second, the frequencies of the waves in the visible spectrum are found to lie between $3.94(10)^{14}$ and $7.63(10)^{14}$ vibrations per second, which corresponds to an interval of about an octave in sound.

According to Langley, the solar spectrum extends in the infra-red to wave-length 0.00027 cm., with a frequency of $1.1(10)^{14}$ vibrations per second; while the radiations from terrestrial bodies below 100° C. extend to 0.0015 cm., with a frequency of $2\,(10)^{13}$ vibrations per second. The shortest measured wave-length in the ultra-violet is 1854 tenth-meters, with a frequency of $1.618\,(10)^{15}$ vibrations per second. These limits are being rapidly extended by contemporary investigations.

In general the frequency of a wave is unchanged when it passes from one medium to another, but its length changes in the inverse ratio of the index of refraction.

The length of any wave in a vacuum may thus be found by multiplying the wave-length in any medium by the index of refraction for that medium.

577. Iridescence. — The peculiar color effect, which depends upon the direction in which a grooved reflector is observed and upon the direction of the source of light, is called iridescence. Mother-of-pearl, which is deposited in smooth, reflective layers, shows this in a marked degree on account of its structure. That it is due to the structure alone may be readily proved by taking a print of a piece of bright mother-of-pearl on white wax, when it will be found that the surface of the wax also shows iridescent colors. Feathers, which have at the same time regular structure and brilliant luster, exhibit the same phenomenon. Most of the beauty of peacock feathers and all that of the iridescent

feathers of the male birds of the turkey, pigeon, and humming-bird families are thus to be explained.

If the source of light is very large, or if the structure of the reflecting surface is somewhat irregular, the colors are less pronounced, or may be wholly wanting, the only effect being a remarkable change of luster with varying obliquity. The effects familiar in white mother-of-pearl, in the gem called cat's-eye, and in satin spar, are of this kind. The variety of feldspar called labradorite, however, has such regularity of fibrous structure that a polished surface illuminated by a source of light of moderate extent will show most vivid colors.

578. Colors of Thin Plates. — The colors exhibited by thin plates are a phenomenon of interference produced, not by breaking the wave surface into limited portions, but by securing a difference of path by reflection from two slightly separated surfaces in a manner precisely the same as in the case of Newton's rings (Art. 545). Thus, if a plate have a thickness of half the wave-length of red light, then the system of waves which are reflected from the second surface of the plate will differ from those reflected from the front surface by a half wave-length in the red, and the two systems will be mutually destructive as regards these particular waves, so that the effect upon the retina would be that of white light minus red, *i.e.* cyan-blue, as will be more particularly shown in Art. 617.

If the plate be a little thinner, say one-half the yellow wave-length, this color would be absent from the combined effect and the plate would appear blue.

Again, suppose the plate to have a thickness of a wave-length of red light, then the retardation of the waves reflected from the back would be $\frac{5}{2}$ wave-lengths, and the

mutual destruction of this particular wave-length would ensue with a corresponding color. This is essentially the explanation of the production of colors by reflection from thin plates, such as a soap film or a thin layer of oil on the surface of water which is not hot.

There is a notable peculiarity of the colors produced in this way which is worthy of consideration, as it greatly affects the character of the phenomena.

It depends on the fact that the colors are not those of the various wave-lengths, as in the case of the prismatic spectrum, or even those of the sum of several wave-lengths, as in the case of many diffraction phenomena, but they are the colors proper to white light after being deprived of one or several definite wave-lengths. Thus, suppose we have a plate of such thickness that the yellow is destroyed in the reflected light, the remainder is blue. But if the thickness is such that the retardation of the second system is $\frac{3}{2}$ wave-lengths of red, it will be $\frac{5}{2}$ wave-lengths of the shorter green waves; hence the color will be white minus such red and green. Now red and green light combined make yellow; hence, white deprived of these hues will be blue, but paler than that produced in the former way, which is called blue of the first order. So a thicker plate would destroy three wave-lengths, and, if the middle one were yellow, would again yield a blue, but paler than either of the others. If this increase of thickness be so far extended as to effect a destruction of the wave-lengths of a series of colors which when combined would produce white, the difference between their sum and white light would also be white. It thus appears that only films of a very few wave-lengths in thickness can produce colors by reflection.

All these consequences of theory can be observed and verified very conveniently in soap films, either by watching a

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the process of tempering.

579. Diffraction ...

images arranged like cells in a honeycomb, though it would appear simply as three systems of equidistant lines crossing the disc under mutual angles of 60°.

More important is the deduction which may be drawn from the indifference as regards the position of the holes before the eye, provided only that the light from no one of them is cut off from the retina by the iris. That this position is unessential follows obviously from the discussion of Fig. 439, where the iris is quite left out of the question. Hence, we might have two systems of holes, of exactly the same size and configuration, so far apart that the light from one system would not materially modify that from the other. In this case the effect would be only to double the quantity of light which forms the image. But the second system of holes need not be remote from the first if there are no new distances introduced except multiples of the original distances. Hence, if a piece of cardboard, such as is used for worsted embroidery, be held before the eye while looking at the artificial star, the effect is the same in kind as though only four of the holes were transparent, though much brighter, and the separate images are smaller.

The phenomena presented by small luminous points seen through fine and regularly woven fabrics such as silk, lawn, bolt cloths, wire cloth, etc., are of this kind.

Many feathers will exhibit beautiful effects in this way. A modification of the phenomenon may be made by holding a perforated cardboard in front of the objective of a telescope directed toward a bright star. In this case we virtually increase the dimensions of the eye and can use correspondingly coarse structure in the screen. Many of the phenomena of diffraction are of surprising beauty, and in all of these, where the image is bright and large, brilliant colors are seen, the explanation of which will be discussed in Chapter XLII.

EXAMPLES.

1. The following observations were made upon the nickel line in the orange of the second spectrum of a diffraction grating ruled with 14,440 lines to the inch. Readings to the right 69° 44′ and 69° 44.5′, readings to the left 345° 41′ and 345° 40′. What was the wave-length of the light corresponding to this line? *Ans.* $\lambda = 5888\,(10)^{-10}$ meters.

2. The deviation of the *F* line in the second spectrum of a diffraction grating was found to be 41° 26.8′. What was the distance between the rulings? *Ans.* $1.468\,(10)^{-4}$ cm.

CHAPTER XL.

DISPERSION.

580. Dispersion. — For simplicity of treatment it has been tacitly assumed that the light-waves which were modified by reflection, refraction, or interference were all of the same kind. In general, however, the light emitted by a luminous point consists of all wave-lengths within a considerable range. In reflection from large surfaces the composite character of the light has no effect on the images formed; but in the case of interference, which depends upon the length of the waves, the phenomenon becomes clearly more complicated, and, though not obvious from what has preceded, the phenomena of refraction are considerably modified. This may be made at once evident by allowing sunlight to pass through a prism. If the light be received upon a distant white screen, it will be found that, instead of a white spot where the deflected light falls, there will be a brilliantly colored strip, the end nearest the original position of the light being red, and that most remote violet. The intermediate colors in order from the red are orange, yellow, green, green-blue, blue, and violet. The change from one of these hues to the next is absolutely continuous, so that the number of colors is limited only by the number of names at our command to designate them. Since the change in direction of propagation of the waves depends only on their less velocity in glass than in air, it follows that those waves which produce the sensation of red move less slowly than those which give rise to the sensation of orange, and than others which are deviated still more. This separation according to wave-

length is called *dispersion*, and the resulting colored image of the source is called a *spectrum*.

Sir Isaac Newton was the first to investigate the phenomenon in a scientific manner and to fix its terminology, using, however, the color names green, blue, indigo, and violet instead of green, green-blue or cyan-blue, and violet, which modern writers have found more appropriate. Newton's discovery of greatest importance was, that after light is thus modified, any one color suffers no further change on passing through another prism. His conclusion was that ordinary white light is compound and made up of an indefinite number of hues, of which seven are recognized by familiar color names. He also showed that if different colors were united, either by allowing them to fall on a concave mirror and reflecting them to a point, or by passing them through a similar prism turned in an opposite direction, the result was light like that from the original source. Thus, both by analysis and synthesis he demonstrated the composite nature of white light.

Newton also found that like prisms of different substances would produce quite dissimilar amounts of dispersion; but fortunately for the development of practical optics, his conclusion that dispersion, which should be regarded as a secondary phenomenon of refraction, increases as the refractive power was an error.

The explanation of dispersion, as due to differences of velocity in the prism, may be corroborated in various ways. Thus, for instance, the experiment described in Art. 545, in which the size of the rings varied continuously as the prism was rotated so as to allow all colors from red to blue to fall on the air film, shows that the length of the waves decreases continuously from the red to the blue end of the spectrum. Or, again, if a thermometer bulb be placed in the sunshine

and viewed through a prism, the artificial star will appear as a fine linear spectrum. If, then, the screen with two needle holes be placed close to the eye between it and the prism, with the line joining the holes at right angles to the spectrum, the spectrum will appear broadened and traversed by a series of fine horizontal lines whose separation diminishes quite uniformly from the red to the violet. The conclusions from these experiments are: 1°, that the velocity of light-waves in glass decreases continuously with decreasing wave-length; 2°, that waves of different length falling on the retina produçe different sensations, the longest waves awakening the sensation of red and the shortest that of violet. The length of the red waves, as shown in Art. 545, is about twice that of the violet.

The change in the index of refraction with the wave-length for a few important substances is shown in the following table.

TABLE OF INDICES OF REFRACTION.

SUBSTANCE.	WAVE-LENGTH IN CM.	INDEX.	TEMPERATURE C.°
Carbon Bisulphide . . .	0.0000589	1.624	25
" " . . .	485	1.648	25
Water	589	1.334	16
"	485	1.338	16
"	434	1.341	16
Rock Salt	589	1.544	24
" "	485	1.553	24
" "	434	1.561	24
Flint Glass	589	1.651	
" "	485	1.665	
" "	434	1.677	
Crown Glass	589	1.517	
" "	485	1.524	
" "	434	1.529	

581. Character of Refracted Images as Affected by Dispersion. — The images formed by the reflection of white light are not affected by its composite character. In the case of refraction, however, the simple geometric image which would be formed by monochromatic light is in general replaced by a spectrum of the object whenever it emits or is illuminated by white light.

Thus, for instance, refraction at a plane surface, as in Fig. 421, forms a virtual spectrum of the source of which the violet end is nearer the refracting surface. The action of a plate is to form a virtual spectrum with the blue end nearer the plate; but this effect will hardly be detected, except in the case of a very thick plate when the image is seen obliquely. Thus, a white pebble seen vertically downward through deep water still appears white, but if observed in a direction away from the vertical, the colors of the spectrum appear very distinctly. So also a prism, instead of forming a virtual image of the source displaced toward the thin edge of the prism, and approached by a fraction of the thickness of the prism, as appeared in Fig. 424, in reality forms a spectrum with the blue end more displaced than the red end, and also brought a little nearer the prism.

582. Dispersive Power. — The value of minimum deviation of light through a prism was shown, in Art. 554, to be

$$\delta = 2i - \alpha.$$

Observing that when the light is incident nearly perpendicularly on a prism of small angle, a condition fulfilled in most lenses,

$$n = \frac{\sin i}{\sin r} = \frac{i}{r} \text{ approximately;}$$

whence $$2i = 2nr = n\alpha,$$

which substituted above gives

$$\delta = (n - 1)\alpha. \tag{1}$$

If n_A and n_H are the indices of refraction for the red and the violet waves respectively, the dispersion, that is, the angular separation between these two rays, may be written

$$(2) \qquad (n_H - n_A)\,\alpha.$$

If n_D be the index of the brightest part of the spectrum, the ratio of the dispersion to the mean deviation, $\frac{n_H - n_A}{n_D - 1}$, is called the *dispersive power* of the substance, and $n_H - n_A$ the *coefficient of dispersion.*

583. Possibility of Achromatism.—Newton, after making experiments on the refractive indices of glass and water, drew the conclusion that the dispersion of all substances was proportional to the deviation, or, in other words, that their dispersive power was constant, and hence that it was impossible to correct the chromatic aberration of a lens by the combination of two or more refracting substances.

Newton's conclusion was disputed, though unsuccessfully, by Euler about a hundred years later. The criticism had the good effect, however, of leading to a more careful study of the phenomena. In this John Dolland met with brilliant success. Repeating an experiment of Newton's with a prism of water opposed to a prism of glass, he found that deviation of light could be produced without accompanying dispersion into prismatic colors. He also found that two varieties of glass, known as crown, or common window-glass, and flint glass, which is characterized by the presence of a considerable amount of lead oxide, possessed very different dispersive powers. Thus, if C (Fig. 443) is a prism producing a dispersion, $(n_2' - n_1')\,\alpha'$, and F is another prism, opposed to the first, and having a dispersion,

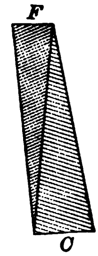

FIG. 443.

$(n_2'' - n_1'')\,\alpha''$, the transmitted light will suffer no dispersion when

$$(n_2' - n_1')\,\alpha' = (n_2'' - n_1'')\,\alpha'', \tag{3}$$

or the angles of the prisms must be inversely as the coefficients of dispersion.

Thus, for instance, if for the flint

$$n_G' - n_D' = 1.66028 - 1.63503 = 0.02525,$$

and for the crown,

$$n_G'' - n_D'' = 1.54165 - 1.52958 = 0.01207,$$

then

$$\frac{\alpha'}{\alpha''} = 2.09.$$

It also follows that a positive lens of crown combined with a negative lens of flint, as shown in Fig. 444, would yield a nearly colorless image. Since the dispersive power of a prism is slightly different for different wave-lengths, it is not possible to secure achromatism throughout the whole spectrum.

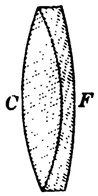

FIG. 444.

It may be noted, in passing, that twenty years before Dolland's success Mr. Chester Moor Hall had invented and made achromatic telescopes; but this fact remained unknown to the world of science until after Dolland's telescopes became famous.

CHAPTER XLI.

MAXIMUM EFFICIENCY OF OPTICAL INSTRUMENTS.

584. Limit of Power of Simple Lens. — It has been shown in Art. 560 that the office of the simple magnifier is to form a virtual image of an object placed near its principal focus at a conveniently greater distance, or, in other words, to render convex wave surfaces flatter after passing the lens. Assuming that this may be done perfectly, the magnification may be increased indefinitely by increasing the power of the lens. An increase of the power, however, requires increased curvature of the lens surfaces, and decreased distance between the lens and its object. For example, a glass lens of spherical form which would make an object appear 100 times larger than it would at a distance of ten inches from the eye would require the object to be within $\frac{1}{30}$ of an inch from the surface of the glass. If the magnification were 1000 times, this distance, called the working distance of the lens, would be reduced to $\frac{1}{300}$ of an inch. It is obvious that this fact would put a practical limit to the useful power on account of the difficulties of illumination and adjustment. The working distance might be considerably increased in special cases by using material of greater refractive power than glass, such as diamond, sapphire, or garnet. In the early part of the century many experiments were tried with these substances, but without the advantages hoped for by their advocates.

There is, however, another source of limitation in power depending on the absolute wave-length of the light-waves. For it is obvious that to increase the power of a lens, since this increase depends on the increase of curvature of the

refracting surfaces in the end, the diameter of the lens must be decreased, and hence the diameter of the wave surface, after passing the lens, must be less with greater magnifying power. Now the experiment on the resolving power of the eye, described in Art. 571, taken in connection with Fig. 438, shows that vision through a hole much less than a sixteenth of an inch in diameter becomes notably impaired because such dimensions are not very great compared to the length of a wave. From this it follows that the minute details of an image could not be recognized if the diameter of the lens is much less than a sixteenth of an inch. This may be rendered clear by considering that, if two points in the object are very close together, the image of these points will appear as a disc of determinate size, and if the separation of these points is no greater than the diameter of the discs, they cannot be seen as two points, but only as one. Increasing the power of the lens will not help, for though it increases the apparent separation of the images, it at the same time, on account of the necessarily diminished diameter, increases the diameter of the disc which represents the image of the point in the same ratio. Experience shows that, when the aperture of the pupil is reduced to a thirtieth of an inch, the indistinctness due to this cause becomes very obvious. Consequently a lens smaller than this in diameter can no longer add to the power of vision, since each point in the image appears as a disc, and each line as a stripe, of which the diameter and thickness increase with the magnification.

It is obvious that the greatest diameter which a lens used as a simple magnifier can have is twice the focal length, since in this case the incident wave surface is hemispherical. Hence, by what precedes, nothing is gained by using a lens having a focal length less than $\frac{1}{60}$ of an inch, that is, by Art. 560, a magnifying power greater than 600, which may

be taken as the theoretical limit. It will be observed that this conclusion is independent of the material of the lens, but that air immersion is tacitly assumed.

The practical limit would probably be found considerably below this; at least, it is tolerably certain that no discoveries have ever been made with simple microscopes magnifying more than 250 or 300 times.

585. Maximum Efficiency of the Telescope. — The result of Art. 568 is of great importance, since it leads at once to the maximum efficiency of a telescope. Thus, calling the diameter of the aperture of the objective D, and the focal length F, by what precedes, the image of any star in the telescope will be a disc whose diameter is $2.4\,F\frac{\lambda}{D}\cdot$ The angular diameter of the image is this quantity divided by F. Assuming the wave-length of light to be $\frac{1}{46000}$ of an inch, the angular diameter of this disc in a one-inch telescope would be $10''.75$, that is, two stars separated by an interval of $10''.75$ would appear to touch, if the light could be traced quite out to the place of the dark ring. In fact, however, the edge of such a disc is so faint that it appears much smaller than this calculation would show, and two stars under most favorable circumstances at somewhat less than half this distance, can just be seen as two distant objects. Consequently the closest double stars which can be seen with a perfect telescope may be found by dividing $4''.56$ by the diameter of the objective in inches.

It is clear that, if the images of two points are not separated, they cannot be made to appear separate by any increase of magnifying power in the ocular, just as, beyond a certain extent, nothing more can be found by magnifying a photograph.

It has thus been shown that the diameter of the objective determines the upper limit of power as well as the lower.

586. Magnification of the Compound Microscope. — The difficulties arising from too close an approximation of the eye and lens to an object may be obviated by the compound microscope, for in this case the eye is removed by a little more than the length of the tube from the object, while a deficiency of power in the objective can be compensated by increased power in the ocular. There is thus no necessary relation either between the working distance or the power of the objective and the total magnification.

If the value of the magnification in Art. 564 be written in the form $\frac{L}{f}\frac{10}{F}$, it may be noticed that $\frac{L}{f}$ is essentially the expression found for the magnification of a telescope, and that $\frac{10}{F}$ is the magnifying power of the objective considered as a simple microscope. Thus, looking simply at the analytical form of this expression, it appears that instead of considering the compound microscope as composed of an objective to form an image, and an ocular to magnify it, it is possible, as far as the magnification is concerned, to regard the objective as forming, at a very great distance, an enlarged image which is viewed by the rest of the instrument used as a telescope. By the aid of this highly artificial but permissible conception it is possible to apply at once results already found for the telescope. In that discussion it appeared that the highest useful power was about thirty times the diameter of the objective aperture expressed in inches. Let this diameter, say D, be expressed in terms of the focal length of the objective by the relation

(1) $$D = 2NF,$$

where N is some number, and F the focal length of the microscope objective. The expression for the useful magnification then becomes

$$(2) \qquad M = 30D \cdot \frac{10}{F} = 30 \cdot 2NF \cdot \frac{10}{F} = 600N.$$

Since, as was shown in Art. 584, the maximum value of D for air immersion is $2F$, the greatest value of N will be 1, and therefore the highest useful magnification for the compound microscope with dry objective will be 600 times, which is the same as that found for the simple microscope.

An important conclusion which may be drawn from the relation $M = 300\frac{D}{F}$ is that the ultimate useful power of a compound microscope depends on the ratio of the effective diameter of the rear surface of the objective to its focal length, and not upon the power of the objective or the length of the tube.

Thus, the efficiency of a perfect microscope is determined by the value of N alone, which may be used to characterize an objective. It is called the numerical aperture.

587. Greatest Resolving Power. — The greatest resolving power of a telescope was seen in Art. 585 to be $4''.56$ divided by the number of inches in the aperture of the objective, or $\frac{4''.56}{2NF}$. Reducing the seconds to circular measure, the linear separation of two points at the distance F would be $\frac{1}{90000}$ of an inch when $N = 1$. That is to say, the finest structure which can be seen in white light ($\lambda = \frac{1}{50000}$ inch) by means of a dry objective is 90,000 repetitions per inch.

Since, as appeared in Art. 585, the resolving power of a telescope increases with diminishing wave-length, the defining power of the microscope will be slightly greater for blue

light. For example, if light of the same wave-length as the F line in the solar spectrum be used, the number rises to 100,000, which may be regarded as the practical maximum for vision. For a photographic plate, however, the defining power is somewhat greater, say by 15 to 20 per cent.

The magnification necessary to exhibit the structure previously mentioned may be determined by the following considerations. A good eye will just recognize a system of lines separated by intervals of from 60 to 70 seconds. Taking the resolving power of a telescope as $\frac{2''.28}{NF}$, the magnification requisite will be

$$\frac{70}{\frac{2.28}{NF}} = 30NF \text{ approximately.}$$

Thus, a total magnification by the microscope of

$$30NF\,\frac{10}{F} = 300N$$

will just enable a keen eye to see the structure, while twice this value will certainly quite reach the limit imposed by the length of the light-waves, even with inferior eyes. This, then, is the meaning of the ultimate useful power being $600N$.

588. Angular Aperture. — The largest angular extent of wave surface which the objective can transmit is called its *angular aperture*. The sine of half this angle, usually denoted by N, is called the *numerical aperture*, and constitutes the true measure of the optical power of a microscope.

589. Hemispherical Front. — Between the years 1850 and 1860 was introduced the practice of making the anterior lens of high-power objectives of a single piece of crown glass nearly hemispherical in shape. Although the value of this

innovation stands almost unapproached, it is impossible to name definitely its inventor. It is usually attributed to Amici, but is also claimed by Wenham.

When the wave transmitted by a lens is of constant curvature, *i.e.* either flat or spherical, the lens is free from spherical aberration, and, leaving out of consideration the chromatic aberration which can be corrected by other means, the resulting image is geometrically perfect. In general, spherical refracting surfaces, the only ones which can be made with precision, do not secure this, and the only practical method of correcting the spherical aberration is to combine convex and concave surfaces, so that the opposite errors shall annul each other as far as possible. There is, however, one special case in which the refraction at a spherical surface is geometrically perfect, and the peculiar advantage of the hemispherical front lens is that it utilizes this case.

Let Fig. 445 represent a piece of glass bounded on the left by a polished spherical surface having c as its center, the line

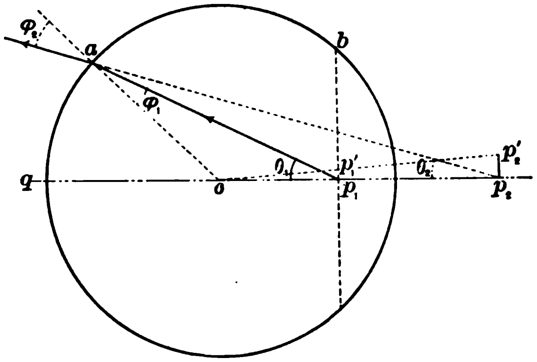

FIG. 445.

cq as the axis of figure, and let p_1 represent a luminous point more remote from the surface than the center. Light-waves diverging from such a point will, in general, lose their spheri-

cal form on refraction; but there is one position of this point for which the waves suffer refraction with considerable change of curvature, but still remain strictly spherical. To find its position, suppose that light proceeding from p_1 in the direction of a has its direction changed so that it appears to proceed from p_2. Call the radius of the sphere R, and its index of refraction n.

Let the angle $p_1ac = \phi_1$, $p_2ac = \phi_2$, $cp_1a = \theta_1$, and $cp_2a = \theta_2$. Then, by trigonometry,

$$\frac{cp_2}{ca} = \frac{\sin\phi_2}{\sin\theta_2}, \tag{3}$$

and

$$\frac{ca}{cp_1} = \frac{\sin\theta_1}{\sin\phi_1}. \tag{4}$$

Multiplying these equations,

$$\frac{cp_2}{cp_1} = \frac{\sin\phi_2}{\sin\phi_1} \cdot \frac{\sin\theta_1}{\sin\theta_2} = n\frac{\sin\theta_1}{\sin\theta_2}. \tag{5}$$

It is obvious that if $\frac{\sin\theta_1}{\sin\theta_2}$ in this equation is constant, the position of p_2 will be independent of the position of a, that is, the direction of the incident light. This condition will be fulfilled if it is possible to set $\theta_1 = \phi_2$ and $\theta_2 = \phi_1$, when equation 5 would become

$$\frac{cp_2}{cp_1} = n^2. \tag{6}$$

Now this condition may evidently be imposed, for since in the triangles cp_1a and cp_2a the angle at c is common, it amounts to making these triangles equiangular. Hence, because these triangles are similar,

$$\frac{cp_1}{ca} = \frac{ca}{cp_2}, \text{ or}$$

$$cp_1 \cdot cp_2 = ca^2 = R^2, \tag{7}$$

which, combined with equation 6, shows

$$
(8) \qquad \begin{cases} cp_2 = nR \\ cp_1 = \dfrac{R}{n}. \end{cases}
$$

Therefore, when both points are defined by equation 8, p_2 is a perfect optical image of p_1 for refraction through the spherical surface to the left of the source.

The construction fails for the surface to the right, since the condition

$$\theta_1 = \phi_2$$

cannot be fulfilled when

$$\phi_2 > \frac{\pi}{2}.$$

The magnification is seen directly from the figure to be

$$\frac{p_2 p_2'}{p_1 p_1'} = \frac{cp_2}{cp_1} = n^2.$$

590. Application to the Microscope.—The preceding theory shows a means by which an object embedded in a sphere of glass, and a short distance from the center, can be replaced by a virtual image n^2 times as great, and absolutely without fault, except so far as dependent on the small variations of n with different wave-lengths.

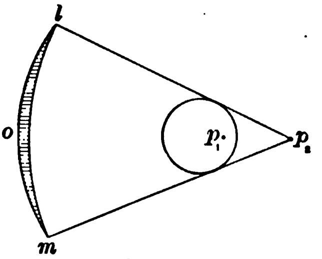

FIG. 446.

The limit of advantage to be obtained in this manner may be derived as follows: Let p_2 (Fig. 446) be the virtual image of the point p_1 magnified n^2 times, and lom an indefinitely thin lens system everywhere equidistant from p_2, which will

render wave surfaces from p_2 exactly flat. In the sense of the previous analysis, the system is just that which, combined with a telescope, will constitute a compound microscope.

To adapt the expression for the highest useful magnification given on page 669 to this case, it is sufficient to notice that the power of this objective system is n^2 times that of lom, or that the maximum aperture of the telescope is $2op_2 \sin lp_2o$.

Comparing Fig. 446 with Fig. 445, this angle is seen to be equal to bp_2c, or cbp_1, and $\sin cbp_1 = \frac{R}{Rn}$. The aperture is, accordingly, $\frac{2op_2}{n}$, or $\frac{1}{n}$ of that considered on page 669. Therefore the power being n^2 times greater, and the aperture $\frac{1}{n}$, the magnification will be n times greater, and the highest useful power $600nN$.

This formula, derived on the supposition that the object is embedded in the lens, still needs a slight modification, for in practice it is necessary that the distance between the lens and the object shall be slightly variable in order to admit of focusing.

In order to fulfill the latter condition, the object O is mounted under a thin cover glass, bb' (Fig. 447), whose influence may be neglected, and the space L between it and the

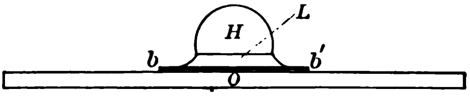

FIG. 447.

hemispherical lens H filled with a liquid. Under these circumstances the waves proceeding from O suffer an additional refraction at the flat surface of the lens, by which their curvature is changed in the ratio of the index of refraction of the

fluid to that of the lens. Hence, the expression for the highest useful magnification becomes

$$600nN \cdot \frac{n'}{n} = 600n'N,$$

where n' is the index of refraction of the immersion fluid.

In the case of dry objectives $n' = 1$ and the limit of resolution, 90,000 lines to the inch, is unaltered. For water immersion, $n' = 1.3$, the limit becomes 120,000, while for a homogeneous immersion, that is, a fluid having the same refractive power as glass, the limit rises to 135,000. In this case the refraction of the cover glass will introduce no error.

Since the greatest known value of the index of refraction for any transparent medium is about 2.5, there is no hope of ever making visible a greater number of lines than 250,000 to the inch. Since the highest powers require the employment of more than a full hemisphere, as appears from Fig. 445, the difficulties of forming and mounting such objectives make them very costly.

It should be remarked that the immersion fluid was originally introduced by Amici, to reduce the loss of light from the face of the objective, and that its theoretical improvement of the resolution was not anticipated before its advantage had been experimentally found.

591. Apochromatic Objective. — The most important advance in recent years in the correction of some of the defects which seemed inseparable from the old construction is the invention by Professor Abbe of what he calls the *apochromatic objective.*

Suppose the objective, which, according to our previous convention, makes a perfect virtual image at an infinite distance, to be divided into two portions as in Fig. 448, the lower of which, *l*, forms a magnified virtual image of the ob-

ject at a finite distance, and the uppermost, *u*, a corrected virtual image of this at an infinite distance.

Now since all transparent substances refract short waves more than long, the image of *o* formed by *l* (Fig. 448 A), assumed to be somewhat under-corrected, will be a complex of colored images from *r* to *b*, the blue image being most remote. The object of the system *u* is to convert all the complex system of wave surfaces, having their centers from *r* to *b*, into plane wave surfaces. In other words, the system *u* must have a higher power for red than for blue waves, and at the same time be nearly free from spherical aberration for all colors. There is no difficulty in meeting these conditions for wave surfaces of moderate angular extent by the combination of negative lenses of flint glass with positive lenses of crown glass; but when the wave surfaces are large, as they must be in powerful objectives, it is found that the power of the upper system is always relatively too small at the margin for the short light-waves.

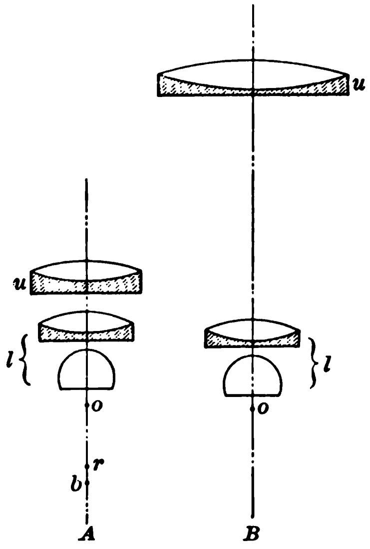

FIG. 448.

The lack of flexibility in the means of correction arises from the fact that in all known glasses a great increase in dispersive power is invariably accompanied by an increase of refractive power. The use of fluids in lenses, though successfully applied by Abbe and Zeiss, is practically debarred on account of the difficulty of keeping them in good order.

A satisfactory correction of the defect mentioned has been secured by Abbe in an entirely different way. If the lens system u, instead of standing as close as possible to l, be removed to a considerable distance, as in Fig. 448 B, the change of curvature which has been produced by u is diminished, and the power of the objective is correspondingly decreased; but at the same time the difference in curvature between the red and the blue is decreased in a much greater ratio. Thus, if the curvature of the red light-waves is half as great in the second case, the difference will be one-fourth as great; or if the curvature is reduced three times by making the distance between l and u twice as great as that between l and r, the difference of curvature will be reduced nine times, and so on.

The new construction consequently admits of varying the relation between the two functions of the system u within wide limits, and yields another arbitrarily variable element for the attainment of a closer correction. This, with a skillful selection of materials to reduce to a minimum the far less serious defect of what is called secondary chromatic aberration, constitutes the important optical apparatus to which the inventor has given the name apochromatic objective.

The construction of this objective entails, however, a defect in the complete instrument which is of considerable interest on account of its general nature. The expression for the focal power of a lens

$$\frac{1}{f} = (n - 1)\left(\frac{1}{r_1} + \frac{1}{r_2}\right)$$

shows that the magnification in a virtual image by the system l is greater for short waves than for long, and consequently the whole objective will magnify a blue object more than a red one, unless the system u exactly reverses this

relation, *i.e.* unless it changes the curvature of the wave surfaces of short wave-length less than those of longer waves, and less by just the proper amount. Now whatever be the case with the system u in Fig. 448 A, it is clear that the cor-

FIG. 449.

responding system in Fig. 448 B cannot correct the defect in question in the anterior portion, because the separation has been made for the sake of reducing the relative difference of curvature of the wave surfaces. Thus, it appears that in this objective and, in general, in any system in which correction for color is obtained by lenses separated by a considerable

distance from uncorrected lenses, the images of an object, though all in the same plane, and therefore achromatic in the ordinary sense of the word, differ materially in magnitude with differing color. This defect produces a characteristic imperfection in the seeing with all high power microscopes. Abbe corrects this defect by adding a compound lens to the ocular, of such construction that it makes its power greater for red light than for blue in the same ratio as the excess of magnification of the objective lies in the opposite direction. These two elements combined, namely, the apochromatic objective and compensating ocular, form the modern perfected microscope.

The accompanying cut (Fig. 449) of an instrument may be regarded as one of the best existing models. The apparatus below the stage is for the purpose of securing at will illumination from any desired direction. It is known as Abbe's illuminator.

CHAPTER XLII.

OPTICAL PHENOMENA OF THE ATMOSPHERE.

592. Optical Phenomena of the Atmosphere. — There are a large number of phenomena, some beautiful, some merely curious, which depend upon the modification which light undergoes in its passage through the atmosphere. In certain cases these phenomena are a necessary consequence of the optical properties of the air alone, or at least dependent upon invariable constituents, as, for example, the blue color of the sky, mirages, and looming, and scintillation of the stars. In other cases the cause is to be sought in bodies temporarily constituting a portion of the atmosphere, as in the rainbow, corona, and halo.

593. Color of the Sky.—If the atmosphere were perfectly transparent, *i.e.* if the light-waves could be transmitted through it without loss, the sky would appear quite black, except where a bright spot marked the presence of a star or a planet. In short, the sky of day would differ from that of night only by the presence of a brighter star, the sun. On the other hand, if the air should only transmit a portion of the wave energy, converting the remainder into heat or some other form of energy unappreciable by the eye, the sky would bear no closer resemblance to its familiar appearance, for it would still remain black, except at points in the direction of the stars where a portion of the light might penetrate.

Professor Langley has demonstrated that what is ordinarily called a clear atmosphere is very slightly opaque to light-

waves. The colored appearance of the sky may be understood from the following reasoning:

Suppose the atmosphere is perfectly transparent, and hence the sky black with a high sun, and that small drops of water are distributed at infrequent intervals, say one to a thousand cubic feet of space. The sky would send light to the eye from all directions, since each drop would scatter the light which fell upon it, though the brightness of the direct sunlight might not be notably diminished. Such a sky would be called *hazy*. If the number of these small drops should be continually increased, the density of the haze would grow, while the brightness of the direct sunlight would progressively diminish until the sky became overcast and the sun invisible.

Now imagine the process reversed; that is, suppose the drops are removed, leaving the distribution fairly uniform until the sky is no brighter than an ordinary clear sky. Then suppose the drops to be reduced in size, but at the same time increased in number at such a rate that the total quantity of light from the sky remains the same. This reduction of size, however, introduces an entirely new element into the consideration, for when it is carried so far that the particles of water have a diameter which is small compared to the wave-length of light, they would be incapable of reflecting these waves, precisely as a floating body on the ocean would be unable to reflect waves whose lengths are great compared to its own dimensions, although it would offer a perfect barrier to the passage of short waves. Before reaching this condition of extreme tenuity, however, a range of dimensions will be passed, which, though small when compared to the wave-length of red light, are not small compared to the wave-length of blue or violet light. Such particles would reflect violet and blue light more copiously than orange

and red light. It is under these conditions, and for this reason, that the clear sky is blue.

It is obviously immaterial of what the particles are made, provided they are small and not coagulated; hence, when light is reflected from a cloud of smoke which is not too dense, especially if the background is black, so that only such light reaches the eye, it appears pale blue. The blue of opal and opalescent bodies has its origin in a similar manner, as has also the color of blue eyes and that of the deep sea.

It is well known to painters that generally a mixture of a white with a black paint gives a strongly bluish gray, although there may be no suggestion of this color in either of the components. This, too, may be explained by a species of selective reflection depending on the minuteness of the reflecting particles.

It follows from these considerations that sunlight which has come through the atmosphere has lost more in short than in long waves, and consequently the hue of such light is somewhat yellow. If the light has passed a very long distance through the air, as when the sun is near the horizon, we may have, with a great diminution in the strength of the waves, a practically complete stoppage of the short waves. This would give yellow, orange, or red, depending on the completeness of the selective action, and also, as will appear in the study of the phenomena of color sensation, to a considerable extent upon the absolute intensity of the light coming to the eye. The colors are strongest after the sun is below the horizon and the light is received only indirectly from reflecting clouds, for the path of the light through the air is much longer than when the source is above the horizon.

An analogous phenomenon is seen in a light smoke, which appears blue against a dark background, as already noted,

but yellow when the background sends more light to the eye than the smoke itself.

The greenish blues and the blue greens, which are not infrequently seen in the sunset sky, are probably always a physiological effect of contrast.

One of the most notable effects of the opacity, or more properly opalescence, of the atmosphere is the change it produces in the aspect of distant objects. Thus, two surfaces, the one light and the other dark, lose something of their contrast as they recede from the eye, the one becoming darker by the absorption of its light through the intervening air, and the other brighter by the superadded light diffused by the air. If the atmosphere is free from coarse dust particles and relatively large particles of water, this added light is blue. Often only a short distance is required to give a strongly blue hue to a sunlit rock, for instance, much less than a mile, while distant hills are always of a strong violet or blue in a clear day. This effect goes under the general name of aerial perspective, and affords the readiest means of estimating the distances of remote objects on land. In a very clear, dry atmosphere it is greatly diminished, so that a moist climate adds greatly to the grandeur of mountain scenery.

594. Atmospheric Refraction.—The velocity of light-waves in air of the prevailing density at the surface of the earth is about three parts in ten thousand less than in a vacuum. In consequence of this retardation taken in connection with the decreasing density of the air at higher altitudes, light entering the atmosphere at its greatest obliquity in equatorial regions has its course changed by about 35 minutes, or not far from the diameter of the sun. The sun therefore appears to be just above the horizon, when, were there no air, it would appear to be just below it.

The effective lengthening of the day is about four minutes at the equator; at higher latitudes the effect would be greater, and in extreme cases it may amount to many hours.

595. Looming. — Occasionally temporary inequalities in atmospheric density give rise to peculiar phenomena of refraction, known as looming and mirage. These phenomena in their simpler forms are not so uncommon but that it is possible to obtain a knowledge of them from observation at least a hundred days in the year.

Let AB (Fig. 450) represent a distant upright object, and SB the level surface of the earth or of water. Now, suppose that the temperature of the air above SB decreases up to

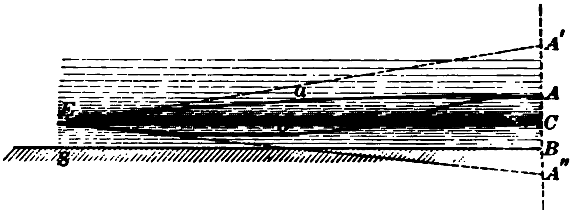

FIG. 450.

the level EC, where it is lowest, and that above this hight it either increases or remains constant. This arrangement, though unusual on a large scale, may obtain under conditions to be noted later. Since the velocity of light-waves in air depends upon the density, and this, in turn, depends upon the temperature and the hight above the surface of the earth, it is evident that only in the direction CE will such waves be propagated without deflection. In all other directions the line of propagation will be a curved one. For instance, a wave surface below CE, and moving toward the left, would tend constantly upward because its upper edge, moving in a denser medium, would advance more slowly than the lower edge.

Likewise the path of a similar wave above CE would curve downwards.

For a discussion of the consequences of these refractions it will be necessary to assume a position for the eye of the observer. Suppose it to be at E. Then the path of that portion of a wave surface having its origin in A would have some such form as AaE, and A would appear at A' in the direction of the tangent at E. Any point between A and C would send out waves whose paths to the eye would be more curved than the one marked; hence they too would be seen above their real position, but raised in a greater ratio than A. Accordingly, AC would be increased in apparent hight, the increase being greater in the lower part, but the width would remain unchanged. This phenomenon is known as looming. It will be noticed that these conclusions are quite applicable to ordinary atmospheric conditions, since in that case the density of the atmosphere decreases from the earth upwards at a diminishing rate. Thus, at a hight of three and a half miles the density is reduced about a half, while to reduce it the remaining half, a further ascent of more than a hundred miles is necessary.

For the reasons mentioned, the hight of a mountain cannot be determined accurately from the distance at which it is visible at sea, nor, conversely, can the dimensions of the earth be found from the measured horizon of a mountain whose hight is known.

596. Mirage. — Light-waves emitted by the point A (Fig. 450) may come to the eye by a path of the form indicated in the line AbE. The apparent position of A would now be at A'' in the direction of the tangent to the curve at E. Any other point of AC will also be a source of wave surfaces, which, becoming distorted by differing velocities of propagation in

different directions, will reach E along two different paths, one lying between AaE and CE, and the other, for the greater part of its course, between AbE and CE. No light from points immediately below C will reach the eye, because, the paths being continually curved upwards until they pass the layer of maximum density, the reversal of curvature there experienced will not be sufficient to make any part of the waves pass through E. Hence, under the conditions supposed, two images will be visible from the point E, one erect and elongated, but in a ratio continuously decreasing from the foot to the top; the other inverted, meeting AC at C, and extending downwards to that point of CB, or of the ground between S and B, which is the first to emit light-waves any part of which passes through the place of the eye. It does not appear that the inverted image is either lengthened or shortened, since no supposition has been made as to the law of change of density; but there will be no alteration of the image in width. It must be borne in mind that the diagram greatly exaggerates the effect of irregular refraction, the changes in direction being in general only a fraction of a degree, instead of many degrees, as in the figure.

If the eye be placed below the level of maximum density, the most notable change would be that the point of the lower portion of the object, or of the ground whose image marks the limit of the inverted image of AC, would be lowered so that more of the region above C would appear in this simulated reflection.

The condition most frequently met in the ordinary phase of mirage, not only of the desert, but also of that extremely common in our own latitudes, is that in which the eye is placed above the level of maximum density. In this case the portion of the ground, or lower part of CB, which can be seen is increased, and the portion of this line which is in-

visible and replaced by an inverted image of AC is correspondingly reduced. Moreover, since the curvature of the paths from A to the place of the eye will, in general, be less, the looming will be less pronounced.

In a broad, sandy plain with a low horizon, which has become heated by the sun, the lower region of the atmosphere may become much warmer than the air a few inches higher, provided there is no wind sufficiently strong to secure a tolerably homogeneous mixing. Under these circumstances there would be a certain region of the plain, at a distance increasing with the hight of the eye, which would be visible, but beyond that only an inverted image of the sky could be seen. The appearance would be virtually that of a smooth body of water, an effect which would be greatly increased should there be any object to break the horizon, such as a tree, for the appearance would then reproduce the familiar reflections from a lake.

In our own regions the conditions most favorable for the production of a lower layer of air of higher temperature than that of the air above are found over considerable bodies of water. For example, in the early hours of clear, quiet mornings late in summer or early in the fall, the atmosphere is often cooled by nocturnal radiation to a temperature considerably below that of the water, which retains much of the heat accumulated during the long summer days. A portion of the heat of the water is imparted to the air in contact with it, and all the conditions that have been considered are then present. The layer of optically rarer air may be only a few inches thick, in which case it is necessary to bring the eye quite close to the surface of the water in order to see very marked effects. There are, however, very few clear, windless mornings in the autumn when it is not possible, in any of the New England bays or harbors, to see distant vessels ac-

companied by an inverted image beneath them. In searching for such phenomena it is often advantageous to employ a small telescope.

Interesting imitations of the mirage may be found by looking at distant objects along a wall, or straight board fence, which has been warmed by the rays of the sun, especially if the surface is moist. The aid rendered by the moisture is due in part to the fact that the velocity of light-waves in a mixture of steam and air is somewhat greater than in air of the same temperature and pressure, but chiefly because the water vapor stops more of the heat radiated from the surface, and thus suffers a greater rise of temperature. Here, of course, the looming is absent, because the air more remote from the wall is homogeneous, but the double image with its limits is very clearly shown.

In high latitudes over fields of ice much more complicated cases of mirage are not uncommon. There it sometimes happens that there is a deep layer of cold air higher up, the eye being far below the region of transition. In this case it is easy to show by a course of reasoning similar to that followed in Fig. 450 that it may be possible to see a distant object through the lower nearly homogeneous layer in its true position with an inverted image above it separated by a greater or less distance, according to the hight of the transition region. Since the upper paths are more curved than the surface of the earth, the inverted image may, under exceptional circumstances, be seen when the object itself is below the horizon, and therefore invisible. This phenomenon seems not to be very rare.

597. Scintillation. — The phenomenon of twinkling, or scintillation, of the stars was first explained by Arago as depending upon the lack of optical uniformity in the atmos-

phere. Several phases of the phenomena may be noted as follows: 1°. It is only the stars that scintillate, the planets appearing quite steady to the eye, with the exception, possibly, of Mercury, which is not only very small, but can only be seen near the horizon, where the conditions are most favorable for the effect. 2°. If the finger be pressed gently against the right eye, so that its image of a twinkling star is slightly displaced from that of the left eye, it may be noticed that these two images flash quite independently of each other; or, if the star be observed through the biprism of Fig. 440, the two images seen with one eye will not change their intensity simultaneously. 3°. If, when a scintillating star is observed through a telescope, the telescope be given a sudden motion by striking it with the hand, a ribbon of light with constantly changing colors may be seen.

All of these observations may be thus explained: Suppose a bright point to be the source of light-waves which come to the eye through the atmosphere, then those wave surfaces which fall upon any area, say one-half of the pupil of the eye or of the objective of the telescope, will in general, on account of the irregularity of the atmosphere, have suffered quite different modifications from those which fall upon the other half. If, now, this difference is such that one portion is retarded any number of half wave-lengths, the two portions will be mutually destructive, and that particular wave-length will be wanting. Should this wave-length be long, the star would appear for an instant greenish; if of medium length, the color would be purplish; and if short, orange or yellow. In a faint star the colors would escape detection without the aid of a telescope, and only variations of intensity would be seen. Anything tending to make the air more homogeneous would reduce the amount of scintillation. It is for this reason that the phenomenon is most

pronounced in a dry atmosphere, and is almost wanting over tropical seas.

It is evident that stars near the zenith should appear more steady than those at lower altitudes, since the light has traversed a minimum extent of the air. Though a single point of a planet may scintillate strongly, since the light from different points of its disc will have suffered very different modifications, it might be anticipated that the average brightness of the sum of all the points would vary but little, which is indeed the fact.

598. Coronas. — Coronas are a series of colored circles seen about the sun or moon, when covered by very light clouds. They are distinguished from the larger circles, called halos, not only by their smaller and variable size, but also by the arrangement of the colors, the inner edge being blue and the outer edge red, which is the reverse order of their occurrence in halos. Fraunhofer showed that coronas may be perfectly imitated by scattering very small, circular, opaque bodies, such as lycopodium powder, in an entirely irregular manner over the surface of glass, and looking through it at the sun or moon.

By the use of a telescope, so as to magnify the effect, and by looking at a much smaller source of light, such as a star or planet, he was able to secure the same result by employing a number of equal discs of tin foil irregularly placed in front of the objective. These experiments show that the phenomenon is a diffraction effect.

It was seen in Art. 573 that when a bright point was observed through a very small round hole it would appear as a disc surrounded by a series of concentric rings having a blue color within and a red without. If another hole of the same size were perforated in the card, it was found that the

disc and rings, now twice as bright, remain, but crossed by a series of dark lines. If the number of holes be increased, retaining the same size, it is found that the disc and rings maintain their positions, but with constantly increasing brightness and complexity of intersecting systems of dark lines. Should the number be increased indefinitely, it may be concluded that the dark lines would become too numerous to be seen, and the final effect would be the same as that of a single aperture of the size chosen, though multiplied in brightness by a number equal to that of the holes. That such is the fact has been established mathematically by Verdet, and may be verified by looking at an artificial star through a piece of tin foil perforated with a large number of small equal holes, or by piercing a sheet of paper with a large number of irregularly distributed holes even as great as one-tenth of an inch in diameter, and looking through it with a telescope at a bright star.

In order to employ these facts in the explanation of coronas, it will be necessary to make use of what is known as the principle of Babinet, which may be thus stated: If illumination occurs at any point on account of the presence of an opaque screen, however complicated, between the source and that point, then, if all the transparent portions of the screen be made opaque, and the opaque portions transparent, the quantity of light at that point will remain unchanged. The proof of this follows from the considerations of Fig. 439. The reason that light appears at p_2, etc., is because the wave surface is limited; consequently the opaque portion of the screen cuts off what would, if added to the waves at p_2, exactly destroy them; in other words, waves of the same intensity, but differing by one-half wave-length in phase. This is indeed but a special case, but the reasoning is perfectly general, and the principle may be accepted as governing all cases.

By means of this principle it is now possible to pass at once from the case of the screen irregularly perforated with uniform circular holes to the glass plate covered with irregularly disposed opaque discs or spheres, and from that to small spheres of water suspended in the atmosphere, since these are essentially opaque, because almost all of the light which passes through them is greatly changed in direction. The essential conditions of a corona are, therefore, suspended particles of water, uniform in size, and so small that the diffraction rings due to them shall be considerably larger in angular dimensions than the sun or moon. As the spheres grow larger, the corona becomes smaller, and *vice versa.* Coronas thus furnish a guide as to whether the droplets of moisture are increasing or diminishing, and an indication of value for predicting changes of weather.

Several other phenomena may be mentioned having an analogous cause and appearance to coronas. Thus, many observers may recognize a system of colored circles surrounding an arc light seen against a dark background, particularly just after awaking from sleep. These become especially marked after a severe blow upon the eye, and remain sometimes for many months with decreasing brightness and increasing dimensions. They are attributed to a slight opacity in the epithelial cells of the cornea.

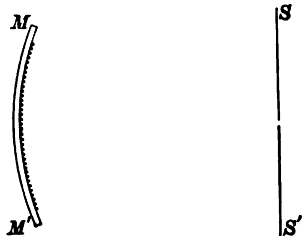

Fig. 451.

Occasionally such circles are to be seen about a light when observed through a sheet of glass upon which there is a considerable deposit of moisture from the air.

An effect not unlike the corona may be obtained by passing light through a small opening in a screen, *SS'* (Fig. 451), and reflecting it from a concave silver-on-glass mirror, *MM'*, so

that the light passes directly back through the hole. If the glass be now breathed upon, so as to tarnish the surface, light which has been diffused at the front surface and then regularly reflected will interfere with light which, first regularly reflected, has suffered diffusion at emergence, and will produce a series of colored rings upon the screen.

599. Rainbow. — Rainbows are produced by refraction of direct sunlight falling on spherical drops of water. In general very little light will be received from such an illuminated drop, if remote, for that which leaves the drop, either after refraction or after reflection from the outer surface, or after having suffered refraction combined with interior reflections, will be sent in every direction, so that little can fall upon the small area of the pupil of the eye. There are, however, certain directions for which this statement does not hold, as may be shown by the aid of Fig. 452. Let o be the center of a drop of water, and AB a plane wave surface moving from left to right and falling upon it. The portion of the wave which meets the drop at a will have its direction of propagation changed and also have its curvature made concave, since the velocity is less in water than in air.

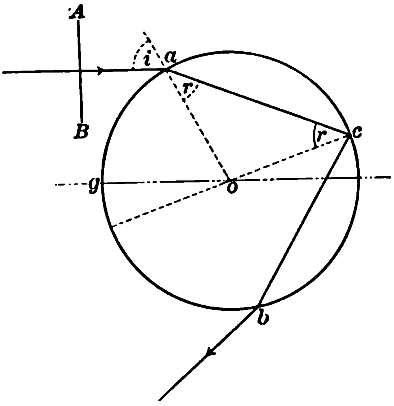

Fig. 452.

Let ac be the new direction of the wave. At c the wave will be reflected, its concavity reduced, and its direction changed to that of the chord cb, which is of the same length

as *ac*, since, by the law of reflection, the angles on either side of the radius drawn to *c* are equal. The wave, accordingly, meets the surface at *b* at the same angle that it left the point *a*, and hence, on emergence, will suffer a change of curvature equal and opposite to that experienced on entering the drop. If, then, the reflection at *c* should change the concave surface into an equal convex one, the emergent surface would be plane.

If the index of refraction of the sphere be less than 2, this condition may always be met by arranging the incidence so that *c* is the center of the refracted wave at *a*. This is not, however, the only case in which a plane incident wave emerges as a plane wave, as may be seen in Fig. 453. Assuming here a larger angle at *a*, there results a greater deviation and change in curvature by refraction, so that the wave surface is concave until it reaches the center c_1; it is thence convex to the first point of reflection, where its convexity is diminished. By a proper choice of the point *a*, this reflection may be made to render the convex waves plane; in which case they would move on without change until they became concave by another reflection, with a center at c_2; thence they would move as convex waves to *b*, where they would be again refracted into plane waves.

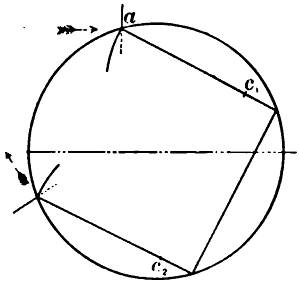

Fig. 453.

Proceeding in a similar manner, it may be shown that there is an indefinite number of ways in which the same result may be attained, the geometrical condition being that, if there is an odd number of interior reflections, the middle one must correspond to a point where the curvature of the

wave surface is infinite, while, if the number of reflections is even, the middle chord, which marks the path of the waves, corresponds to a region where the curvature is zero.

The analytical conditions may be found as follows: Let the angle of incidence at a be called i, and the angle of refraction r. The deviation, or change in direction, of the wave between its entrance at a and emergence at b may be written down at once from the figure

(1) $\delta = i - r + \pi - 2r + i - r = \pi - 2(2r - i),$

or, after m reflections,

(2) $\delta = 2(i - r) + m(\pi - 2r) = m\pi - 2\{(m+1)r - i\}.$

Suppose that when i is increased by a small amount, ϕ, r increases by the amount ψ. Substituting these values, the deviation for the slightly altered angle of incidence becomes

(3) $\delta' = m\pi - 2\{(m+1)r - i + (m+1)\psi - \phi\}.$

In order that the emergent wave shall be plane, it is necessary that portions of the wave entering very near a shall emerge with the same deviation.

Comparing equations 2 and 3, it appears that this will be the case when

(4) $\phi = (m+1)\psi.$

To find the value of i corresponding to this condition, draw two circles (Fig. 454) whose radii have the ratio of the index of refraction n. Let $pov = i$, and draw pu parallel to ov. The intersection of pu with the outer circle determines the angle

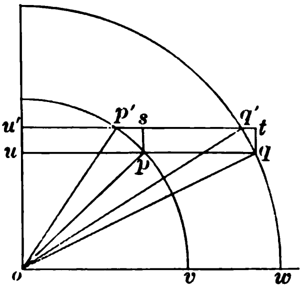

FIG. 454.

$$qov = r,$$

because, by construction,

$$\sin pov = n \sin qov.$$

Similarly, drawing $u'q'$ near and parallel to uq,

$$pop' = \phi, \text{ and } qoq' = \psi.$$

Draw ps and qt perpendicular to ov. Then, because their sides are perpendicular, the triangle psp' is similar to puo, and qtq' to quo; therefore the angle

$$p'ps = i,$$

and

$$q'qt = r.$$

Hence

$$\frac{sp}{pp'} = \cos i \text{ and } \frac{tq}{qq'} = \cos r.$$

Observing that

$$sp = tq,$$

$$pp' \cos i = qq' \cos r;$$

but

$$pp' = op \cdot \phi, \qquad qq' = n \cdot op \cdot \psi,$$

and

$$\phi = (m + 1)\,\psi;$$

whence

(5) $$(m + 1) \cos i = n \cos r.$$

Combining this with

$$\sin i = n \sin r,$$

we have, finally,

(6) $$\cos i = \sqrt{\frac{n^2 - 1}{m^2 + 2m}}.$$

This condition of a stationary value of the deviation is also one that determines either a maximum or a minimum devia-

tion. It is easy to see in this case that it is not a maximum, for when the incident light passes through the center of the drop

$$m = 1 \text{ and } \delta = \pi,$$

which is greater than that of equation 6.

Substituting the indices of refraction for water, the following values of the minimum deviation are found:

m	δ Red.	δ Violet.
1 . .	138°	140°
2 . .	231	235
3 . .	309	313
4 . .	403	410

600. Formation of the Bows. — From the preceding table it is seen that all raindrops upon which the sun shines at angular distance between 42° and 40° from the antisolar point, that is, 138° to 140° from the sun, will appear bright, and consequently a portion of a ring will be seen against the sky, if the sun is less than 42° above the horizon. So, too, drops at a distance of from — 51° to — 55° from the antisolar point will form a similar bright bow. The minus sign in this radius signifies that the light received from any drop in the secondary bow has entered the lower instead of the upper side, as in Fig. 453. The secondary bow is necessarily much fainter than the primary, because its light has suffered two partial reflections, and this condition must hold with stronger reason in bows of higher order. This fact is, without doubt, sufficient to account for the absence of the fifth bow, which is 486° from the sun, or of a radius of 54° about the antisolar point. Were the third, or even the fourth, as favorably situated for observation, it would certainly be visible at times. As, however, these bows are only about 50° from the

sun, and that portion of the sky is always strongly illuminated by light which has been transmitted through the drops, it is not remarkable that they have never been seen.

It follows from the table of deviations that the primary bow is red on the outside and violet on the inside, while in the secondary bow the order of these colors is reversed. The prismatic colors cannot be very pure on account of the angular dimensions of the sun.

Since the deflection of the light which forms the bows is a minimum, it follows that no drops nearer the sun than 138° can send any light to the eye by a single interior reflection, and none nearer than 231° can send any light by two interior reflections; therefore those drops which are situated between the bows, being nearer in one direction than the first limit, and nearer in the other direction than the second, can send no light of either modification. This accounts for the relative darkness of the sky between the two bows, which forms a notable feature of a well-developed rainbow.

601. Supernumerary Bows. — Since the light which produces the bows comes from portions of the plane incident waves which have suffered a minimum deviation, it follows that there are parts of these waves above the most effective portions which have the same deviation, such light coming from drops just under the primary bow, or just above the secondary. But such portions will have passed through different lengths of water, and will therefore be in a condition to interfere.

It is easy to see that this will produce a series of repetitions of each color, just inside the primary and outside the secondary, of rapidly diminishing brightness and at an angular distance decreasing with increasing size of the drops. These bands, known as supernumerary bows, are most often

seen under the highest part of a bright inner bow, and, more rarely, as an accompaniment of the outer bow.

The conditions of distinctness, namely, uniformity of size and smallness of the drops, are such as are more likely to obtain at a higher altitude.

Another effect of interference is to change slightly the dimensions of the bows, decreasing the apparent diameter of the inner and increasing that of the outer bow. This change, though small for the ordinary rainbow, may amount to several degrees in the white bow, of which the description follows.

Occasionally a very bright primary bow is seen, with only a tinge of red on the outside and of blue on the inside. It is formed when a bright sun shines on a dense wall of mist, where the droplets are sufficiently large to give a tolerably definite reflection, but at the same time differ greatly among themselves as to size. It is always smaller in angular diameter than the colored bow, for the reasons just stated.

The lunar rainbow appears almost colorless, on account of its faintness, just as foliage loses nearly every trace of color by moonlight. This is, however, merely a physiological effect.

Sometimes arcs of rainbows are seen on a sward when covered with dewdrops, or in a spray thrown up by the bow of a boat. In such cases, although the image on the retina is always a circular arc, yet, since the neighboring objects permit an estimation of the distance of the drops from which the light proceeds, we ascribe to the bow the form of the projection of this circle on the surface. Hence, if the sun has an altitude greater than 42°, the bow will appear as an arc of an ellipse; if less, as an arc of a hyperbola.

The history of the theory of the rainbow is an interesting one. Descartes gave the first geometrical theory in 1637,

without, however, accounting for the existence of colors. This explanation was added by Newton in 1704. The explanation of supernumerary bows was suggested by Young in 1804, but first completely worked out by Airy in 1836.

602. Halos. — Very frequently a circle of about 22° radius, red inside and bluish without, and showing indistinctly some of the intermediate prismatic colors, is seen in a slightly hazy sky to surround the sun or moon. On rare occasions this is accompanied by complicated series of curves and bright areas, some of which may be vividly colored and others quite colorless. Such curves are called halos, and the limited bright areas are called parhelia, or sun-dogs. These are all attributable to minute crystals of ice floating in the atmosphere, and their theory is pretty completely worked out — at least for all the common features. Thus, the ordinary circle of 22° radius is explained as follows: A minute crystal of ice is known to take, generally, the form of a right hexagonal prism, of which any two alternate prismatic faces would form a 60° prism. From the measured index of refraction of ice it may be demonstrated that such a prism would have a minimum deviation of about 22°. If, now, we imagine the air filled with a host of such little crystals floating in it, we may easily see that all of them which are at nearly this angular distance from the sun, and which at the same time chance to have approximately the proper orientation, will refract light towards the eye, while none of those nearer the apparent position of the sun can do so, since the angle of 22° is a minimum angle. On the other hand, crystals more remote from the sun than these can also divert light towards the eye, but in view of the fact that to do so requires a very delicate adjustment of orientation, the proportion of those thus favorably situated must very rapidly

decrease with increasing angular distances from the sun. These considerations lead us to expect a sudden increase in the luminosity of the sky, when such crystals are abundant, at a distance of 22° from the sun, followed by a pretty rapid reduction as we go outwards. When regard is paid to the different refractive power for different wave-lengths, the cause of the color of the circle is evident.

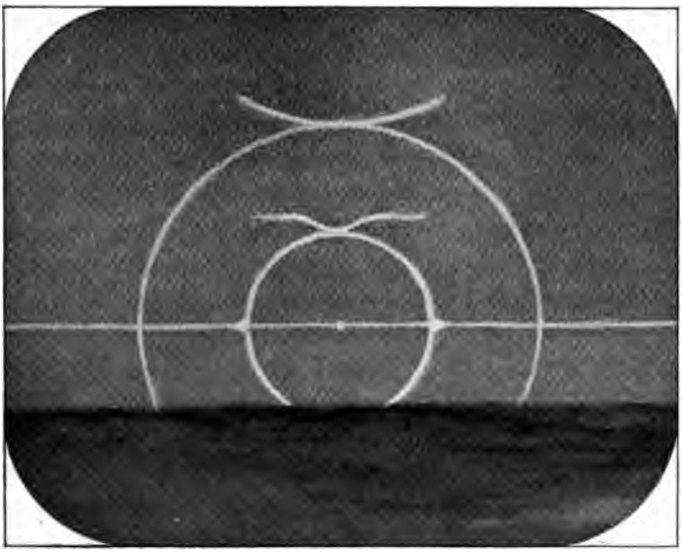

FIG. 455.

The dihedral angle of 90°, formed by the sides and bases of the hexagonal prisms, also produces a circular halo in a similar manner, but of 46° radius. This is necessarily much fainter than the 22° circle, and is in fact very infrequent.

The other phenomena, some of which are represented in Fig. 455, are also due to suspended crystals of ice, but to those which have a prevailing constancy of direction of their axes. That small bodies of a definite form would, in falling

through a quiescent atmosphere, assume a general likeness of direction is at least probable. At any rate, an assumption of this kind explains all of the more common complications of halos. Thus, crystals having their crystalline axes vertical cause the familiar sun-dogs, which are almost as common as the smaller circle, and the tangent arc to the outer circle. To such crystals also is ascribed the horizontal colorless band passing through the sun, sometimes extending completely around the heavens, called the parhelic circle; but to explain the tangent arcs to the 22° circle, and a number of other rarer features, we must assume the presence of crystals with their axes horizontal.

CHAPTER XLIII.

RADIATION AND ABSORPTION OF LIGHT-WAVES.

603. Selective Absorption. — Although the vibrations of light- and heat-waves must be regarded as electrical in their ultimate nature, they obey a law of emission and absorption quite analogous to those already found for the purely mechanical vibrations of sound-waves, in that if the body is capable of vibrating so as to emit waves of certain definite periods, then whenever waves of these periods fall upon the body they will be absorbed. The character of the absorption which takes place in any substance may be conveniently studied by analyzing the transmitted light with a prism, in which case the spectrum appears crossed by dark bands corresponding to the missing waves. Thus, the spectrum of light which has passed through a piece of red glass shows that it transmits the red waves copiously and some of the orange, but that the green and blue are entirely absorbed. Likewise a piece of cobalt glass will be found to cut off all the bright red, the orange, and yellow waves, but to transmit all of the violet and blue with some of the green, and, singularly enough, a band of dark red very near the end of the spectrum. A solution of chlorophyll, the green coloring matter of plants, exhibits dark bands in the red, yellow, green, and violet, and a dilute solution of permanganate of potash shows several characteristic bands in the green. If a piece of red glass be heated till it becomes luminous, and then placed in a dark room, it will be seen to emit a bluish green light, that is, waves of those periods which it absorbed

when cold, thus illustrating the proportionality of its emissive and absorptive power.

These phenomena are more strikingly exhibited in gases on account of the freedom of the molecules to vibrate in their own peculiar periods. Thus, if the light emitted by incandescent sodium vapor at a moderate temperature be analyzed by means of a prism of high dispersive power, it will be found that the sodium emits only waves of two definite wave-lengths in the yellow. On the other hand, if the light from some incandescent solid be allowed to pass through a sodium-tinged flame at a lower temperature than that of the first source, say through the flame of an alcohol lamp with salted wick, it will now be found that the spectrum is crossed by two black lines situated exactly at the place previously occupied by the bright sodium lines.

604. Kinds of Spectra. — Spectra are often classified for convenience as *continuous* and *discontinuous*. The former are emitted by incandescent solids or liquids, and are characterized by the presence of all waves from the red to a higher limit determined by the temperature.

Discontinuous spectra are marked by the absence of particular waves, and are of two sorts, the absorption spectrum arising from the passage of white light through an absorbing medium, and the spectrum emitted by an incandescent gas.

605. Spectroscope. — The spectroscope is an instrument conveniently arranged for analyzing and observing the light emitted by various sources.

The disposition of parts in a form often used in the chemical laboratory is shown in Fig. 456. The prism, *P*, used to form the spectrum, rests on a table, *A*, at the center,

supported by leveling screws so that it may be readily adjusted. On either side of the prism, and movable about the vertical axis of the table, are placed a telescope, *T*, and a tube, *C*, known as the collimator, furnished at the outer end with a narrow slit, *S*, and at the inner end with a positive achromatic lens. The light which it is desired to examine is first concentrated on the slit. The waves diverging from this point have their curvature reduced to zero by the lens of the collimator. Falling on the prism at the

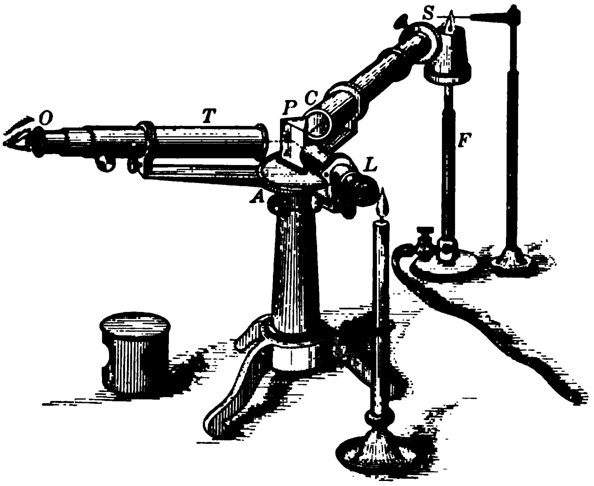

FIG. 456.

angle of minimum deviation, they suffer dispersion, and after passing through *T* appear to diverge from points in the field of the telescope, which differ with different wavelengths. For purposes of comparison a bright scale of equal parts in *L* is often so arranged that its image by reflection is made to coincide with the spectrum; or a portion of the slit is illuminated by light from some standard source, and the other portion with the waves to be investigated, in which

case the spectra appear placed one above the other. When the instrument is furnished with a graduated circle to meas-

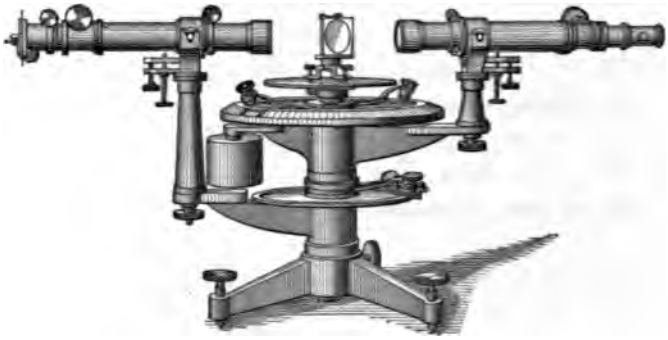

Fig. 457.

ure the deviation produced in passing through the prism, it is called a *spectrometer*, of which a simple form is shown in Fig. 457.

606. Spectrum Analysis. — It has been found that each element in the gaseous state emits a discontinuous spectrum, differing greatly in complexity, but always so characteristic that there is not even a remote resemblance of one to another. Since, also, it is found that at the temperature of gaseous incandescence the spectrum of a compound consists of the sum of spectra of the constituent elements, the spectroscope furnishes a valuable aid to qualitative analysis. Since it requires but a very small quantity of a substance to yield its characteristic spectrum, a number of rare elements that had previously escaped detection have been discovered by the spectroscope. Among these may be mentioned caesium, rubidium, thallium, indium, gallium, scandium, ytterbium.

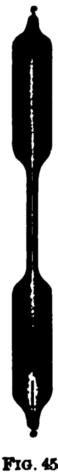

Fig. 458.

In the study of gases the electric discharge through

an exhausted tube, such as that shown in Fig. 458, is usually employed.

607. Solar Spectrum. — The spectrum of sunlight was first observed by Wollaston, in 1802, to be crossed by a number of dark lines. These lines were independently discovered[a] and carefully studied fifteen years later by Fraunhofer, after whom they are usually named the Fraunhofer lines.

The more prominent have been designated by the letters of the alphabet, as in Fig. 459.

The corresponding wave-lengths in air are given in the table on p. 652.

Fig. 460 is a copy of a photograph of a portion of the solar spectrum, nearly all of which lies beyond the accepted visual limit. The lettering of the prominent

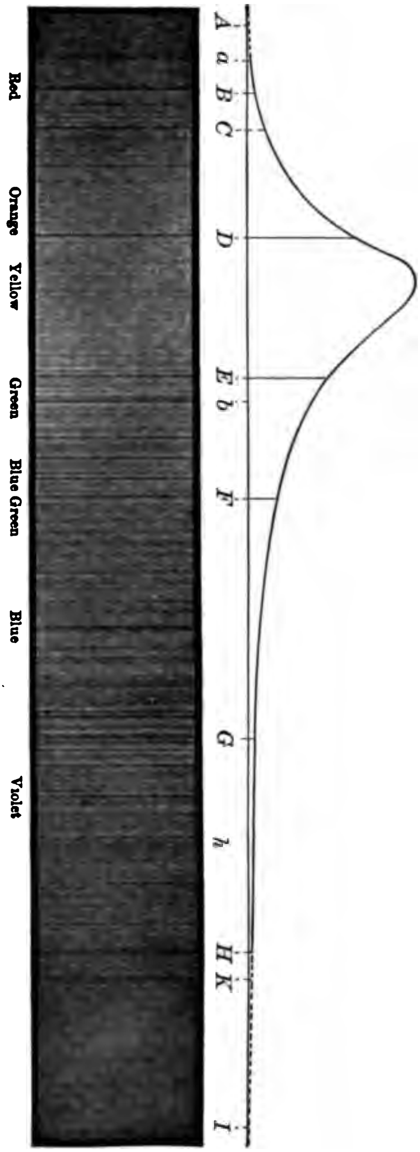

Fig. 459.

lines has been extended from Fraunhofer's system. The obvious explanation of the Fraunhofer lines is that the principal light of the sun is emitted from an incandescent central mass (photosphere), and that the missing waves are absorbed either by the gaseous envelope about the sun (chromosphere) or in our own atmosphere. Nearly all the lines of the solar spectrum have been found to be identical with those of the emission spectra of known elements, from which it may be certainly concluded that these substances are present in the sun. Sodium, hydrogen, iron, calcium,

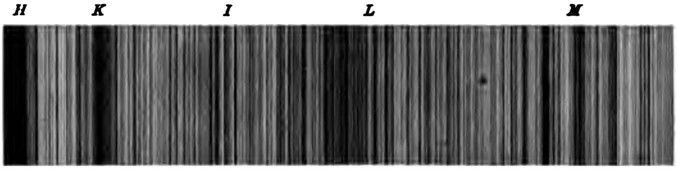

FIG. 460.

lithium, aluminium, titanium, chromium, carbon, manganese, nickel, copper, magnesium, with perhaps thirty others, undoubtedly exist. Antimony, arsenic, gold, boron, bismuth, mercury, have not as yet been identified, but it by no means follows that these substances are absent. On the other hand, it is of interest to note that an unknown bright yellow line in the chromosphere, which has been for many years provisionally assigned to a substance called helium, has recently been identified with the spectrum of a gas obtained from the rare mineral cleveite.

A and *B*, in Fig. 459, are oxygen lines due to absorption in the earth's atmosphere. The double *D* line is produced by sodium, and the group *b* by magnesium. *C*, *F*, and *h* are due to hydrogen.

The preceding principles may be applied to the stars, and their chemical nature inferred in the same manner. Many nebulae and a few stars emit bright-line spectra.

608. Displacement of Lines. — The principle which was explained in Art. 495 has several important applications in the case of light. It was shown in the article mentioned that when a source of waves was approaching an observer at a velocity, u, the frequency was increased in the ratio $\frac{u}{v}$.

If, now, u is quite small with respect to v, a condition always fulfilled in the case of light, the wave-length, since it varies inversely as the frequency, would be decreased in the same ratio; that is, by the amount $\frac{u}{v} \cdot \lambda = \frac{u}{n}$. Since the molecules of a luminous gas are in motion with all velocities between certain limits, say $+u$ and $-u$, it follows that every line of the spectrum must possess a finite breadth, although it corresponds to a single frequency of vibration.

The displacement of known lines in the spectrum has also furnished a means of estimating the proper motion of fixed stars in the line of sight, the speed of rotation of the sun about its axis, and the velocity of the gases in a solar eruption.

609. Absorption. — *Body Color.* The experiments described in Art. 603 show that when a body exerted selective absorption the color of the body was the result of the combination of those waves which were allowed to pass, so that a piece of yellow glass was yellow, not because it colored the light, but because it absorbed the blue waves.

Likewise if waves penetrate a short distance into a substance and are then reflected out, its color will be that of the light it transmits. The color so determined, which is that of most natural objects, is known as *body color.* When, for instance, a piece of blue glass is placed in front of a piece of yellow glass, obviously that light will pass the combination which is transmitted by both, namely, green. This accounts

for the fact that a mixture of blue and yellow pigments appears green, which is not at all the color of a mixture of yellow and blue light.

Sometimes the amount of absorption for different waves varies with different thicknesses. In this case the color of a thick layer of the substance may differ much from that of a thin one. For instance, glass colored with a cobalt salt appears blue in a thin plate, but violet in a thick one.

Surface Color. Certain substances, notably such metals as gold and copper, and many of the aniline dyes, exert the power of selective reflection; that is, they completely reflect certain waves in addition to the amount sent back in ordinary reflection, and transmit others. The light which is transmitted by thin films of such substances is complementary (Art. 613) to that which is reflected.

610. Anomalous Dispersion. — The law found in Art. 574 showed that in a diffraction spectrum the deviation of the different wave-lengths was equal to an angle whose sine varied directly as the wave-length. By inclining the grating to the direction of the incident light the deviation from the normal to the grating may be made so small that it may be taken as proportional to the wave-length. Such a spectrum is called a *normal* spectrum.

In the case of dispersion through a prism, however, the separation of the Fraunhofer lines in different parts of the spectrum is not alike for any two substances. In fact no substances have yet been found which will afford coincidences between more than two lines in each spectrum (Fig. 461). Hence complete achromatism cannot be produced by any combination of two lenses.

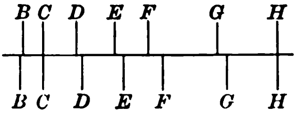

FIG. 461.

In vacuous space the velocity of light appears to be independent of the wave-length. When, however, light enters ponderable media, since the waves are not indefinitely great compared to the elementary structure, it might be anticipated that the velocity of the waves and the period of free vibration of the ether would be considerably modified, and probably according to a complex law. In substances without selective absorption, *i.e.* those whose molecules do not have natural periods approaching that of the waves, the velocity is an increasing function of the wave-length or period of the light under consideration. A form of this function given by Cauchy, and found to agree moderately well with observation throughout a considerable range, is

$$n^2 = a + \frac{b}{\lambda^2} + \frac{c}{\lambda^4} + \cdots,$$

where a, b, c, etc., are constants, and n is the index of refraction.

When, however, the substance is capable of selective absorption, *i.e.* possesses natural periods of molecular vibration which are the same as those of certain of the light-waves, the deviation of the light on the lower or red side of the absorption bands is increased and that on the upper side decreased in an abnormal manner. This phenomenon, known as *anomalous dispersion*, is very strikingly exhibited by an alcoholic solution of fuchsine, one of the aniline dyes. By transmitted light the solution shows a strong absorption band in the place of the green, which is entirely lacking. Anomalous dispersion is exhibited by all substances which show surface color.

The arrangement and intensity of the colors in the spectrum, as produced by a strong fuchsine solution in a hollow prism of about 1° angle, are shown in Fig. 462. The red,

orange, and yellow have their usual order, but the violet is deviated less than the red and separated from it by a dark space. The position and length of the dispersion spectrum produced by pure alcohol in the same hollow prism are marked by the dotted lines.

Ketteler has given a dispersion law, containing four constants, which not only accords with the observed optical

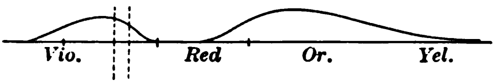

FIG. 462

properties of transparent media, but also with substances possessing anomalous dispersion. The characteristic feature is a constant wave-length corresponding to each absorption band in the spectrum of light transmitted through the substance, and for a single band takes the form:

$$n^2 = -k\lambda^2 + a + \frac{D\lambda_o^2}{\lambda^2 - \lambda_o^2},$$

where λ_o is the wave-length of the absorption band.

611. Re-emission, or Emission of Absorbed Energy. — In general some portion of the energy which has been absorbed by a body, when ether-waves fall upon it, is afterward emitted by the body, but with a change in the period. The commonest case is that in which light-waves falling on a body, such, for example, as black carbon, are transformed into heat and afterward in part emitted as waves too long to affect the retina. In certain cases, however, the transformed waves lie within the range of vision and give rise to the phenomena of luminescence. When, for instance, a beam of sunlight is allowed to pass through a green solution of chlorophyll, the path of the light is marked by a bright red

streak. In a piece of glass colored with oxide of uranium the track appears greenish yellow, and in kerosene oil a faint blue. Such substances are called *fluorescent.*

A solution of sulphate of quinine is affected only by waves of very short period. Such a solution will emit a bluish light when placed some distance beyond the upper end of the visible spectrum.

In the substances mentioned the fluorescence continues only an instant after the radiations have ceased to fall upon them. In certain other substances, notably the sulphides of calcium, of strontium, and of barium, the luminous waves continue to be emitted for a considerable time after the stimulating cause has been removed. This phenomenon has been called *phosphorescence.* It differs from fluorescence only in duration.

CHAPTER XLIV.

SENSATIONS OF COLOR.

612. Spectral Colors. — Whenever light-waves from a white body are systematically arranged in order of their wave-length either by refraction in a prism or by diffraction at a grating, the normal eye perceives a luminous band, the colors of which succeed each other in an entirely continuous manner, showing that light of each wave-length within the visible spectrum corresponds to one definite color sensation.

Theoretically, the best definition of a few standard colors would be in terms of the wave-lengths of light which produce those sensations. Thus, the six spectral colors for which we have common names might be designated as follows:

				Line
Violet	4190	to	4240	*g*
Blue	4670	"	4720	
Green	5140	"	5190	*b*
Yellow	5770	"	5820	
Orange	6060	"	6110	
Red	6560	"	6610	*C*

The difficulties of reproducing these colors in pigments render such definitions less important than they appear.

613. Sensations of Color. — Although, as has just been seen, each light-wave produces a definite color sensation, it is not at all true that a given color sensation is produced by only one determinate set of waves. Thus, the analysis of the light emitted by a white-hot solid shows that the combined

effect of all visible light-waves is capable of producing the sensation of white. But it is also possible by taking only two kinds of waves to produce a sensation of white, which is in no way distinguishable from that produced by all waves. Any two colors so related are termed *complementary.*

Similarly, it is possible to combine three, four, or any number of colors to produce the sensation of white. In an analogous manner, any given color sensation may be produced in a variety of ways by the combination of two or more other colors.

It must be carefully noted, however, that these examples apply solely to the combination of different colored lights, and not at all to the mixtures of pigments, which exercise a differential, rather than an additive effect on light. (See Art. 609.)

Experiment shows that the smallest number of colors by the addition of which all other color sensations may be produced is three. Although the selection of the particular triad of colors in which the others shall be expressed is arbitrary, except that they must not be too close together in the spectrum and shall produce white by addition, the phenomena of abnormal color vision have favored the choice of red, green, and blue.

614. Maxwell's Color Discs. — A very convenient method of studying the sensations produced by the combination of lights of different colors is to observe a disc divided into colored sectors and whirled rapidly, when, on account of the persistence of visual impression, each color is spread over the whole area of the circle. This method was first employed by Newton to show that white light might be produced by the union of what he called seven primary colors, but was later extended to the systematic study of color vision by

Maxwell, who used discs of colored paper slitted radially and overlapped, so that they could be made to expose a sector of any desired extent. A smaller disc of the color whose composition is desired is placed at the center, with sectors of black and white, if necessary, and the proportions of the fundamental colors altered until, when the whole is whirled, the outer ring appears as a satisfactory match to the center.

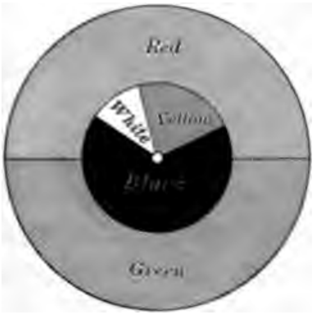

FIG. 463.

Fig. 463 shows a particular case in which it was found that 51 parts of red and 49 parts of green gave a good match to 20 parts of a certain yellow mixed with 72 parts black and 8 parts white, which may be symbolized by

$$51\,R + 49\,G = 20\,Y + 8\,W + 72\,b.$$

This relation is known as the color equation of the particular yellow under examination.

615. Characteristics of a Color Sensation. — The complete definition of a color sensation requires the determination of three independent characteristics known as ***hue***, ***saturation***, and ***luminosity***.

By hue is meant the color name proper. The three hues in terms of which all others are expressed are red, green, and blue.

The natural hue of any object, as has been explained in Art. 609, depends upon the combination of waves from the incident light which the body is able to return to the eye. Bodies which diffusely reflect all the visible waves in equal ratios are termed white, and those which reflect none, black.

Obviously, the colors of objects may vary greatly with the nature of the incident light, but the color names usually associated with objects are always to be understood as those perceived by a normal eye, when these objects are illuminated by a source of white light.

By saturation is meant the freedom of the color from white. The same idea is often expressed by the words "purity" or "vividness." Many surfaces reflect a considerable amount of white light in addition to their proper colored light. They may be described as pale, and are very far from saturation. The colors of the spectrum are examples of saturated colors.

By luminosity is meant the brightness of the color, or the intensity of the sensation. The luminosity of two colors of the same hue may be taken as proportional to the energy of the light-waves received by the eye, but the intensity of the sensation produced by waves of different length varies in a complex manner with the energy received, and with the observer. The intensity curve for spectral colors, as determined by Fraunhofer, is shown in Fig. 459.

616. Photometry. — For the purpose of comparing different sources of light it is usual to define the intensity of the light at any point as the energy per unit area which passes the point. Suppose I is the intensity of the light at a point whose distance from the source is d, and suppose a sphere to be drawn about the source as a center and passing through the point; then, if E is the total outward flow of energy through this surface in a given time, the intensity will be

$$I = \frac{E}{4\pi d^2}.$$

Similarly, if another sphere be drawn at a distance, d', then, since the amount of energy which passes both surfaces is the same, the intensity at any point in the second sphere will be

$$I' = \frac{E}{4\pi d'^2},$$

whence

$$\frac{I}{I'} = \frac{d'^2}{d^2}, \tag{1}$$

or, in general, the intensity at any point varies inversely as the square of the distance from the source.

The usual method of comparing one light with another is to arrange the two sources at such distances that they appear

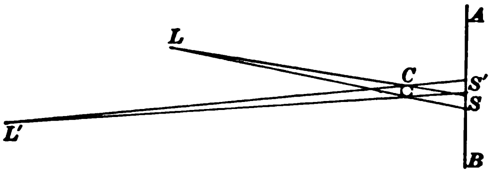

FIG. 464.

to produce equal illumination upon a screen. In practice there are a great variety of ways of arranging the apparatus, but as the measurement of intensity neither demands nor indeed admits of great precision, a description of two simple forms will sufficiently illustrate the process.

In the method commonly ascribed to Rumford, a cylindrical rod, C (Fig. 464), is placed before a screen, AB, and the lights L, L', arranged at such distances that the portions of the screen S, S', which are in partial shadow appear to be equally illuminated. The relative brightnesses are then calculated by equation 1.

The brightness of any source is usually expressed in terms of an arbitrary unit called the candle-power, which is defined

as the illuminating power of a spermaceti candle $\frac{7}{8}$ inch in diameter, and burning 120 grains an hour.

Another method of measuring the intensity of any source of light, devised by Bunsen, employs the principle that the greased surface of a piece of white paper will appear darker than the surrounding portion when viewed by reflected light, but brighter when observed by transmitted light.

By illuminating each side by different sources of light, equal intensity of illumination may be judged with considerable accuracy from the uniform brightness of all parts of the paper. In order to make a simultaneous observation of both sides, the screen is placed in the angle between two mirrors, M', M'' (Fig. 465), so that its images, S', S'', may be compared

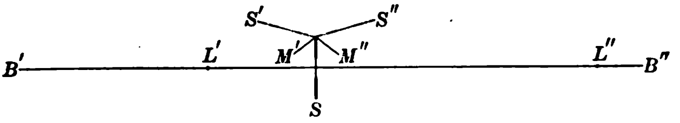

FIG. 465.

by the eye, which is placed directly before S. The system, consisting of the screen and mirrors, is attached to a bar, $B'B''$, along which it may be slid at pleasure, in order to adjust the distance from the lights L', L''.

None of these methods are capable of yielding very satisfactory results if there is a difference of color in the sources of light, for the intensity of stimulation of the nerves is a function of the wave-length as well as of the energy received.

617. Color Diagram. — In making a systematic study of all colors, some scheme of orderly classification is of primary importance. In the arrangement most frequently adopted, and known as Maxwell's color diagram (Fig. 466), the hue

and saturation are determined by the position of a point in a plane.

The three fundamental colors, red, green, and blue, chosen for the discs, and, as nearly as possible, of the same luminosity, which may be taken arbitrarily as unity, are placed at the corners of a triangle. Then any color which, for example, is matched by combining 6 parts of red with 4 parts of green, is located at a point, O, which divides the line RG so that

$$\frac{GO}{OR}=\frac{6}{4};$$

or, in other words, at that point which would be the center of gravity of the components considered as weights.

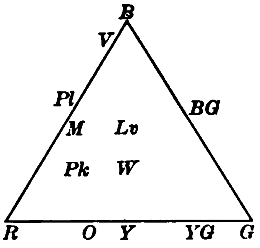

Fig. 466.

Examining in this way the various possible mixtures taken in pairs, it will be found that orange, yellow, and the yellow-greens must be placed along the line RG; crimson, purple, and violet on the line RB; and the series of colors for which we have no other names than blue-green and green-blue on the line BG.

If the three discs are now arranged in such proportion that when their colors are mixed by rotation they produce white or a gray, which is to be regarded as a white of low luminosity, the center of gravity of these three numbers, regarded as weights, will mark the position of white, which will be very near the center of the triangle.

Since, by the mode of placing the hues already mentioned, the position of the combination of any two of these hues may be found on the line joining them by the familiar center of gravity rule, it follows that the area within the triangle is occupied by a combination of the colors on the edge with white, or, in general, that the hue of any color is determined

by the angular position of a point with respect to the red, green, or blue, and the saturation by its distance from the central white. Since three dimensions are not available in a plane diagram, the third characteristic, luminosity, is denoted by a number written over the point.

By the aid of such a diagram it is possible to designate every known color with precision, and also to predict the result of mixing any two or more hues. It should, however, be remarked that it would be clearly impossible to imitate, by the use of colored papers, the saturation and luminosity of many natural objects.

The complementary of any given color may be read at a glance from the diagram. Thus, for example:

Vermilion	and	Blue-Green
Orange	"	Greenish Blue
Yellow	"	Blue
Greenish Yellow	"	Violet
Green	"	Purple

618. Non-Spectral Colors. — It will be noticed that quite a number of colors which have received distinctive color names do not occur in the spectrum.

SATURATED.	NON-SATURATED.	
Purple	Straw	Lilac
Magenta	Sage green	Pink
Brown	Lavender	Flesh Color, etc.

Of these, the only ones that are saturated are the purples, magentas, and the browns. The first two arise from a mixture of red and blue, while the last are produced by simply reducing the luminosity of the colors at the lower end of the spectrum, that is, red, orange, or yellow.

Several familiar non-saturated colors may be defined approximately by the following equations:

Lavender	$= 75B + 25W$
Lilac	$= 75B + 20R + 5W$
Pink	$= 50R + 25B + 25W$
Magenta	$= 50R + 50B$
Olive	$= 45G + 30R + 25b$

619. The Young-Helmholtz Theory of Color Sensation. — The theory which up to the present affords the most satisfactory explanation of the phenomena of color sensation was proposed by Young in 1802, but having since been considerably extended by Helmholtz, it usually bears the name of its joint authors. Its fundamental assumption is that the nerve termini of the eye are of three distinct sorts, which when stimulated give rise to the sensations of red, green, and blue, respectively. The stimulation by different wave-lengths of light, of each of these sets of nerves, which may for convenience be styled the red nerves, the green nerves, and the blue nerves, is shown graphically in Fig. 467, where the abscissas determine the wave-lengths, and the ordinates of each curve represent the excitation of the nerve for that wave-length. The maximum for the first set is just below the orange, for the second set in the yellowish green, and for the third set just above the blue. In general, light of a single wave-length produces a simultaneous excitation of all three sets of nerves. Thus, light near the D line stimulates the blue nerves feebly,

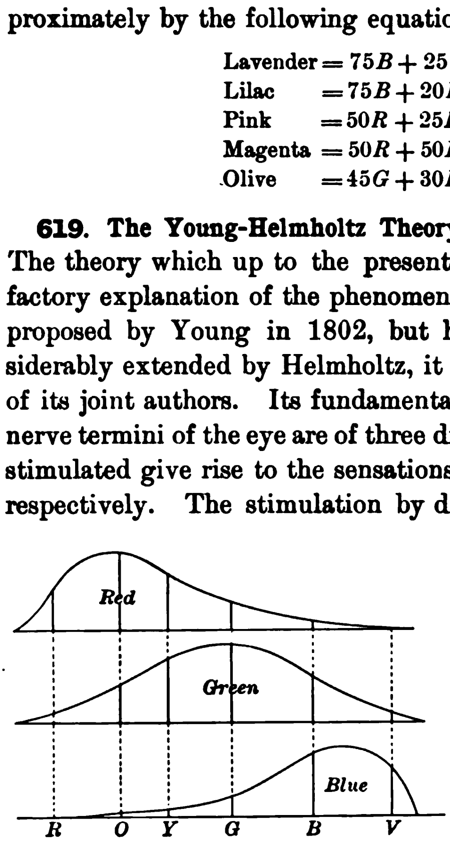

FIG. 467.

the green nerves strongly, and the red nerves nearly as much, giving a resulting sensation, yellow. When all three sets are stimulated to the same degree, the impression received is white. Since no wave-length produces a sensation into which a single set of nerves enters alone, it follows that no color will appear perfectly saturated. It is further evident from the figure that the saturation of a mixture of red and violet, *i.e.* purple, is nearly as great as that of the spectral colors, and that green is the least saturated.

When the intensity of the light is so low as to produce a very feeble excitation of the nerves of the eye, it is not possible to distinguish the impression of one wave-length from that of another, and all appear alike gray.

620. Abnormal Color Vision. — Strong evidence in favor of the Young-Helmholtz theory is to be found in the simple explanation which it affords of the phenomena of defective color vision.

In certain persons the red perceptive nerves appear to be lacking. Consider, for instance (Fig. 468), the character of the sensation experienced by a red-blind person, *i.e.* one who entirely lacks the red perceptive fibers. The red waves produce a moderate stimulation of the green nerves, but the effect on the blue nerves is very feeble. These waves accordingly appear to such a person as a green of low luminosity and strong saturation. Passing up the spectrum, the green of maximum saturation and considerable intensity will be reached at g, about halfway between Y and G. Beyond this point the intensity increases somewhat, but the color becomes paler till white is reached at w.

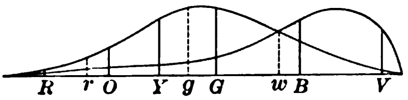

Fig. 468.

The hue of shorter waves is bluish, but the saturation increases with diminishing wave-length. Two waves, such as *r* and *g*, may appear to a red-blind person alike in hue and intensity, but a normal eye would recognize the shorter waves as having a lower intensity.

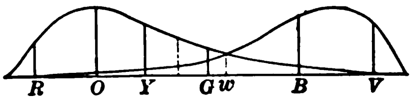

FIG. 469.

The visual sensations of a green-blind person, that is, one lacking the perceptive nerves for green, may be predicted in a simple manner from Fig. 469. All waves from red to green produce a sensation of red with a maximum intensity near the spectral orange, but of gradually diminishing saturation. At *w* the sensation is white or gray, beyond which the saturation and luminosity both increase. If a green-blind person succeeds in distinguishing red and green, it is usually by the aid of the different intensities of the two sensations.

The color sensations of the blue-blind are exhibited by Fig. 470. In this case there is evidently no confusion of red and green, but the spectrum will appear of low saturation throughout the first half, with a white band in the yellow. All waves beyond this produce a sensation of green of diminishing intensity and increasing saturation. A feeble violet would hardly be distinguishable from black.

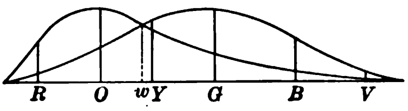

FIG. 470.

Abnormal color vision of analogous peculiarity would obviously be produced if one set of nerves, instead of being entirely lacking, was considerably enfeebled with respect to the others. All these defects are classed under the general head of color blindness. Its prevalence is found to be about 4 per cent in men and $\frac{1}{2}$ per cent in women.

621. Detection of Color Blindness. — Holmgren's method of testing for color blindness is as follows: A large variety of colored worsteds, including red, orange, yellow, yellow-green, green, blue-green, blue, violet, purple, pink, brown, and gray, with several variations of luminosity and saturation in each hue, are placed in a pile upon a table. A sample skein of pale, pure green is laid at one side of the pile, and the subject instructed to place with the sample all other skeins which at all resemble it, without reference to a match. If in this selection any reddish or bluish hues, or a gray, are classed with the green sample, the person is color-blind. In order to decide the nature of the defect, a sample magenta skein is shown, and a selection of resembling colors from the pile requested. If in this second test blues or violets are classed with the magenta, the person is red-blind; if gray or greens are associated with the magenta, he is green-blind; if red or orange are placed alongside the magenta, he is blue-blind.

622. Insufficiency of Young-Helmholtz Theory. — This theory of color sensation represents very satisfactorily the conditions of ordinary vision, and is therefore of great value and interest; but it certainly fails in the case of feeble visual impressions, and must be in some way modified if completeness is to be hoped for. Probably the prevailing tendency to this end is the assumption of a fourth sensation, independent of color, which may be called the *sensation of brightness*, and which is supposed to respond to a stimulus much too feeble to awaken the sensation of color. With this addition it is possible to conceive of complete color blindness without insensibility to any portion of the spectrum, which seems to be the condition of the extreme marginal portions of the retina. There are even some reasons for supposing

that the cones of the retina are alone concerned with sensations which may be described in the language of the Young-Helmholtz theory, while the rods are the means for producing this sensation of brightness.

EXAMPLES.

1. A gas flame placed at a distance of 96 cm. from the grease spot of a Bunsen photometer is found to give the same illumination as a standard candle at a distance of 31 cm. What is the candle-power of the flame? *Ans.* 9.6 C. P.

2. Two sources of light, of 2 and 32 candle-power respectively, are placed 180 centimeters apart. Where must a screen be placed on the line joining them in order that it shall receive equal illumination from each source? *Ans.* 36 cm., or − 60 cm. from smaller light.

CHAPTER XLV.

POLARIZATION.

623. Polarization of Light. — When light-waves exhibit different properties in different directions at right angles to the line of propagation, the light is said to be polarized. Since in a longitudinal wave it is impossible that there should be any difference between one side and another, it follows that light-waves must be of the transverse type.

If the paths of all the points in a wave surface are alike, the light is perfectly polarized. When the paths are straight lines the polarization is called plane, but they may be ellipses or circles, in which cases the polarization is called elliptical or circular polarization, respectively, with the added term of right or left, if it is convenient to designate the direction in which the particle moves in the ellipse. In the case of plane polarized light the plane of polarization is defined as the one which is perpendicular to the direction of vibration. When a train of light-waves is reflected or refracted at the surface of a transparent medium, such as glass, it suffers, in general, a more or less complete polarization, as may be shown by reflecting from a second surface, when the intensity of the reflection will be found to depend upon the relative azimuths of the two planes of reflection.

The explanation is that the vibrations of the light which was reflected from the first surface were reduced for the most part to a direction parallel to the surface, and those of the refracted light, similarly, were confined to a direction perpendicular to the plane of incidence, or, in other words, the reflected waves are polarized in the plane of incidence,

and the refracted waves at right angles to it. The amount of the polarization, which is the same in both the reflected and refracted trains, depends upon the angle of incidence, and reaches a maximum when the direction of the refracted ray is at right angles to the reflected ray.

From this it follows that the angle of refraction is the complement of the angle of incidence, or calling the latter, the angle of polarization, α,

(1) $$\tan \alpha = n.$$

For glass this angle has a value of 57° or 58°. Polarization, even at the polarizing angle, is not quite complete in isotropic substances except for those having a value of $n = 1.46$.

If a beam of polarized light be allowed to fall at the polarizing angle upon a second surface of glass (Fig. 471 *a*) held

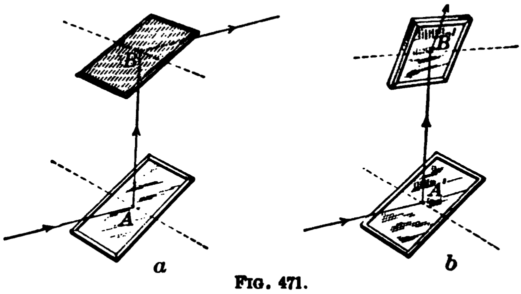

FIG. 471.

parallel to the first, the light will all be reflected and none refracted; but if the second glass be revolved about the direction of the incident light as an axis, through a right angle (Fig. 471 *b*), no light will be reflected, all the waves being refracted. The first mirror in this experiment may be termed the polarizer, and the second the analyzer.

624. Refraction in Non-Isotropic Media.—When a disturbance enters a medium which possesses different properties in different directions, in general the original disturbance is

propagated as two systems which correspond, respectively, to the fastest and the slowest modes of vibration of the particles in the line along which the disturbance moves. A mechanical illustration of this principle may be seen in a rod of elliptical or oblong cross section. If any portion of this rod be displaced transversely in a direction not coincident with either of the principal axes of the cross section, and suddenly released, two waves will run along the rod, one in the plane containing the major axis of the section, and the other at right angles to it, and each traveling with a velocity peculiar to that mode of vibration.

In a somewhat analogous manner, when a train of light-waves falls upon a crystalline substance, the disturbance within the medium is propagated as two sets of waves, polarized in planes at right angles to each other and giving rise to the phenomenon of double refraction, presently to be described in detail. That the transmitted light is polarized may be shown by placing two slices of tourmaline cut parallel to the axis in a beam of light. If these slices are laid one upon the other in the same relative position, a certain amount of light will pass both sections; but if the slices are rotated till they cross at right angles (Fig. 472), the light which passed the first will be entirely stopped at the second. In this particular crystal one of the two refracted trains of waves mentioned above is so rapidly absorbed that it does not emerge from the first section.

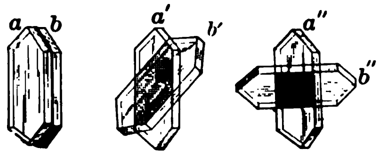

Fig. 472.

625. Double Refraction of Light. — The phenomenon of double refraction was discovered in 1669 by Bartholinus,

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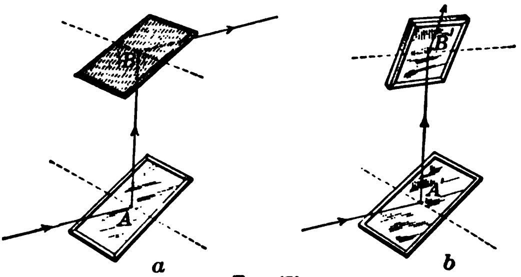

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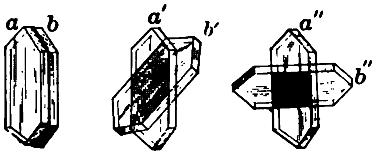


FIG. 472.

625. Double Refraction of Light. — The phenomenon of double refraction was discovered in 1669 by Bartholinus,

who found that when a single incident beam fell upon a crystal of Iceland spar (calcite) it was divided into two beams, one of which obeyed the ordinary law of refraction, and the other a complex and hitherto unknown law. The two systems of waves are, accordingly, distinguished as the *ordinary* and *extraordinary* waves.

Double refraction occurs in all homogeneous anisotropic media, but is most conveniently studied in crystallized minerals.

In every doubly refracting crystal there is at least one, and in many two directions, called optic axes, in which bifurcation of the ray does not occur. In accordance with this principle, all crystals, except those of the cubic system which are singly refracting, are classified in two systems as *uniaxial* or *biaxial* crystals.

626. Wave Surfaces in Uniaxial Crystals. — The explanation of the phenomenon of double refraction in uniaxial crystals, given by Huyghens soon after its discovery, was that any disturbance in such a medium spread as two wave surfaces, which, determined by the principle now bearing his name, were for the ordinary waves a sphere and for the extraordinary waves a spheroid (Fig. 473). Each of these surfaces has an axis in common which is coincident in direction with the optic axis; when a train of plane waves falls upon such a crystal, the refracted waves are also plane and may be found as follows. Let AB (Fig. 474) represent the surface of a uniaxial crystal and pC the direction of the optic axis, taken for simplicity in the plane of the paper. Also, let pp_1p_2 be the plane incident wave perpendicular to the plane of the paper. While the wave-front is traveling the distance $p_2p'_2$ in the air, suppose the spherical disturbance originating at p has spread in the crystal to Co and that the spheroidal wave

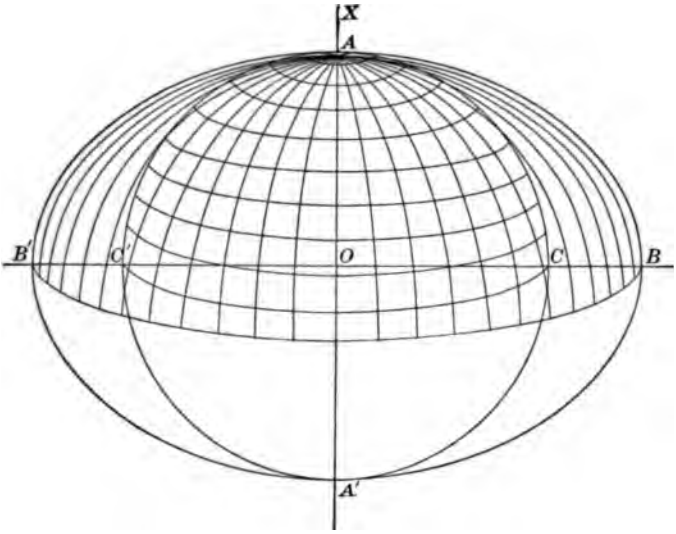


FIG. 473.

has reached Ce. Likewise, in the same time, let the disturbance in the wave-front at p_1 have moved to p'_1 in the air and to o' and e' in the crystal. Then, by Huyghens's principle the new wave-fronts corresponding to pp_2 will be the envelopes of the elementary disturbances, which in this case are seen to be planes $p'_2o'o$ and $p'_2e'e$, tangent, respectively, to the sphere at o, to the spheroid at e, and passing through a line at p'_2 perpendicular to the plane of incidence. Confining the

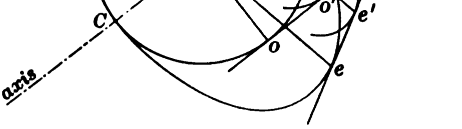


FIG. 474.

attention to a small pencil of light incident in the direction ip, it is seen to be divided into two refracted pencils, — the ordinary po, which is perpendicular to the wave-front in the plane of incidence, and obeying in every respect the ordinary law of refraction, and the extraordinary pencil pe, drawn from the point of tangency e of the plane through the perpendicular at p'_2. pe is, in general, neither perpendicular to the wave-front $p'_2e'e$, nor does it lie in the plane of incidence, though it happens to in the figure, because the optic axis is taken in the plane of incidence. In one particular case, however, when the plane of incidence is perpendicular to the optic axis (Fig. 475), the extraordinary ray is both perpendicular to its wave-front and in the plane of incidence. For this case the velocity of the extraordinary wave bears a constant ratio to the velocity in air for all angles of incidence, its value being known as the extraordinary index of refraction.

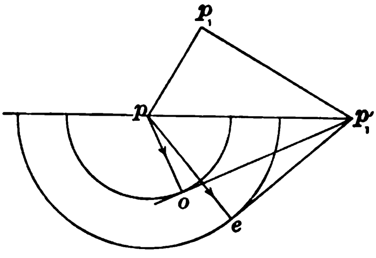


Fig. 475.

Thus, denoting $p_1p'_1$, the velocity in air, by v, the velocity po of the ordinary wave by v_o, and the velocity pe of the extraordinary wave by v_e, the two indices of refraction are

$$n_o = \frac{v}{v_o},$$

$$n_e = \frac{v}{v_e}.$$

Those crystals in which $n_e > n_o$ are called ***positive***, and those in which $n_e < n_o$ are called *negative*. In the former, of which ice and quartz are examples, the ellipsoid lies within the

sphere; in the latter, as for example in Iceland spar and tourmaline, the ellipsoid lies without the sphere (see Fig. 473).

The ordinary ray is polarized in a plane passing through the optic axis, and perpendicular to the refracting surface, and technically known as a principal plane; the extraordinary ray is polarized in a plane at right angles to this.

627. Pile of Plates. — Since the ratio of the velocities of light at emergence is the reciprocal of that at entrance, it follows from Brewster's Law, equation 1, that the angle of polarization for incidence in the denser medium is the complement of that in the rarer. Hence, if light fall on a plate of glass at the polarizing angle, the refracted portion will strike the second surface also at the polarizing angle, and the portion reflected at this surface will be polarized. Likewise, if there be a series of plates parallel to the first, the transmitted ordinary light will meet each successive surface at the polarizing angle, and the reflected portions will all be polarized in the same plane. This arrangement, known as a "pile of plates," forms quite an efficient and inexpensive polarizer. To secure satisfactory results, it is necessary to employ at least a dozen layers of thin plate glass.

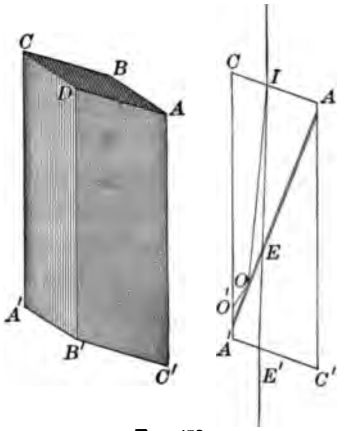


FIG. 476.

628. Nicol's Prism. — The most effective means of producing a beam of polarized light is by double refraction. In order to secure one train of waves free from the other, Nicol devised the prism shown in Fig. 476. An elongated

rhomb of calc-spar is cut by a plane passing through the obtuse angles A, A' and parallel to the diagonal DB. The faces of the section are then polished and cemented together again by means of Canada balsam.

Now, this substance has an index of refraction less than that of the ordinary wave and greater than that of the extraordinary, so that the extraordinary wave will be transmitted without sensible change, while the ordinary wave is totally reflected and hence removed from the field.

629. Foucault's Prism. — Foucault's prism (Fig. 477) differs from Nicol's in the substitution of an air film in place of the layer of Canada balsam. Since the critical angle for the ordinary ray in Iceland spar is about 37° 14′, and for the extraordinary ray 42° 23′, it is evident that the angle of incidence on the section must be intermediate between these values in order that the extraordinary ray shall be transmitted, and the ordinary totally reflected.

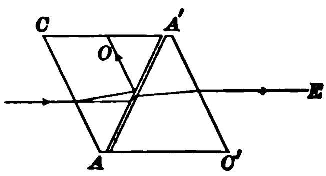


Fig. 477.

This construction is not only more easily made, but also requires less than two-thirds as much material as is necessary for a Nicol's prism, a matter of considerable importance in the present scarcity of Iceland spar. There is, however, a considerable loss of light by reflection from the air film and the angular field is much smaller.

630. Hartnack's Prism. — The usefulness of the ordinary Nicol's prism in connection with a lens system is considerably impaired by the loss of light reflected from the oblique ends, by the influence of imperfections in these surfaces upon the definition on account of their inclined position, and by

the small angular limits of the field, which are determined by the indices of refraction and the position of the section plane.

Fig. 478 shows an improved form of polarizing prism devised by Hartnack and Prazmoski, in which they have remedied the first two defects by cutting the ends of the prism *PQ* and *RS* so that the light enters and leaves them normally.

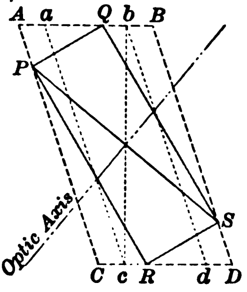


FIG. 478.

By giving the section plane *PS* a position perpendicular to the optic axis, and cementing the two halves with linseed oil, which has an index of refraction identical with the extraordinary index of Iceland spar, the field is increased from less than 20° to 35°.

The size of the rhomb necessary to cut such a prism is shown at *ABCD*, while the size necessary for a Nicol of the same thickness is shown at *abcd*.

631. Double Image Prisms. — By using a prism of a double refracting substance, two images of a bright source of light may be obtained with the transmitted beams polarized in planes at right angles. When a single prism is employed, it is cut so that the refracting edge is parallel to the optic axis, because the difference of deviation between the ordinary and the extraordinary rays is then greatest. The dispersion may be corrected by a reversed prism of glass placed in front.

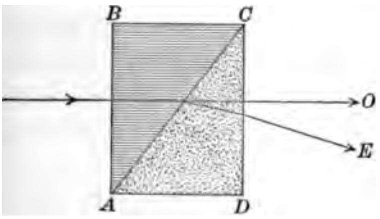


FIG. 479.

Rochon varied this construction by making the front prism

of the same material as the back one, but with the face AB (Fig. 479) perpendicular to the optic axis. Light incident on the face AB perpendicularly is not modified until it reaches AC, where double refraction occurs; the ordinary ray continues without deviation, since the index of refraction is the same in both parts, while the extraordinary ray is bent toward the base or the edge of ADC, according as the crystal is positive or negative.

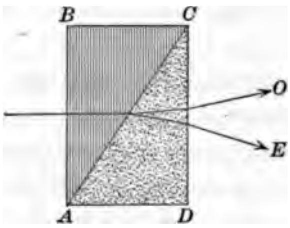


FIG. 480.

A greater deviation may be secured by the construction shown in Fig. 480, due to Wollaston, in which the face of the front prism is cut parallel to the optic axis, with the edge A perpendicular to it. In this case, although there is no deviation in the first prism, the waves are divided into two systems, with vibrations respectively parallel and perpendicular to AB, which traverse the crystal with different velocities. On entering the ACD, the planes of the vibrations do not change, but that system which before had the greater velocity will now have the less, and *vice versa*. Thus, each system will be deviated by nearly equal amounts, but in opposite directions.

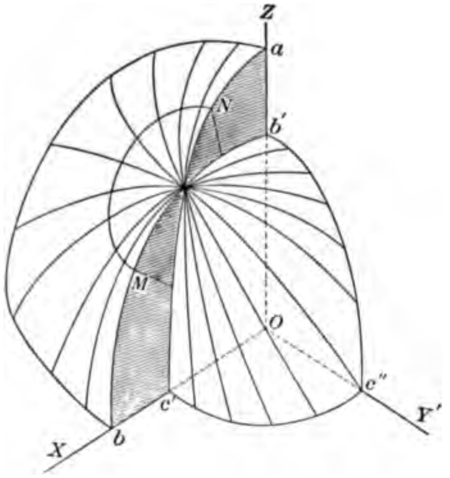


FIG. 481.

632. Wave Surface in Biaxial Crystals. — In biaxial crystals

neither of the refracted rays, in general, obeys the ordinary law of refraction, but there are two directions in which both refracted rays coincide. The phenomena were first explained by Fresnel on the assumption that a disturbance in such a crystal was propagated as a wave surface of two sheets whose form might be given by an equation of the fourth degree, a portion of which is shown in Fig. 481.

The intersections of this surface with the co-ordinate planes are shown in Fig. 482. aa', bb', and cc' are arcs of circles, and $b'c$, $c'a$, and $a'b$ are arcs of ellipses. OM, the perpendicular to the common tangent MN, and a line symmetrically placed on the left side of OZ are the optic axes, for in this direction of propagation only one plane wave-front exists.

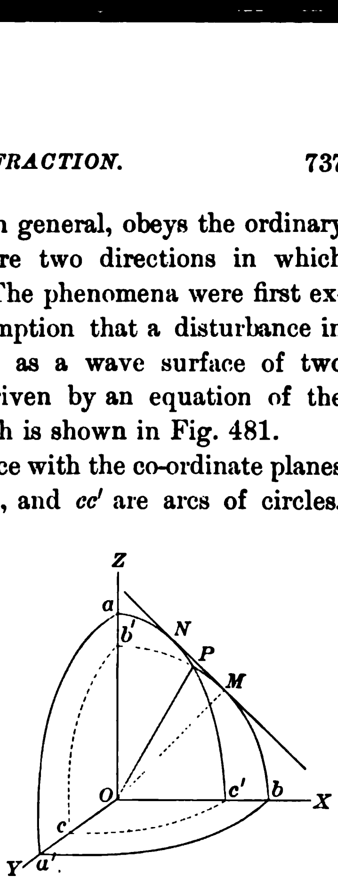


Fig. 482.

The position of the two refracted plane waves resulting from an incident plane wave may be found by the same construction as that employed in Fig. 474, namely, by drawing through the trace of the incident plane two planes tangent to Fresnel's wave surface. The lines drawn from the center of the surface to the points of tangency give the direction of the refracted rays.

633. Conical Refraction. — It was pointed out by Sir William Hamilton that if the refracted plane wave should have such a position, MN (Fig. 482), that it was tangent to both sheets of the wave surface, this plane would touch the surface in a circle, and any line drawn from O to this circle of contact would be a possible direction of the refracted ray.

The truth of this prediction was tested by Lloyd in the following experiment.

A piece of aragonite, CD (Fig. 483), was cut with its faces perpendicular to the bisector of the optic axes, and a sheet of metal foil pierced with a small hole, O, attached to the upper face. A pencil of light was admitted through a hole, S, in the screen AB, and received on a second screen, FG. On changing the position of the crystal slowly, until the proper angle of incidence was secured, the two images of S suddenly spread out into a ring, showing that the path of the light through the crystal was the surface of the cone OMN. This case is distinguished as internal conical refraction.

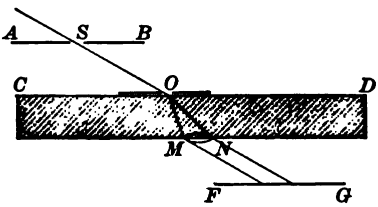


FIG. 483.

A second peculiarity, likewise suggested by Hamilton and verified by Lloyd, relates to the direction OP (Fig. 482). By the construction of Art. 626 this would be the direction of the ray corresponding to any one of the infinite number of plane waves which could be drawn tangent to the surface at P. Thus, if light were to fall upon the crystal in the direction of any one of the elements of a certain cone, it would be doubly refracted at the surface, so that one of the rays would in every case coincide with the singular direction OP. On emerging from a second surface parallel to the first, these rays should again be refracted so as sensibly to regain their original

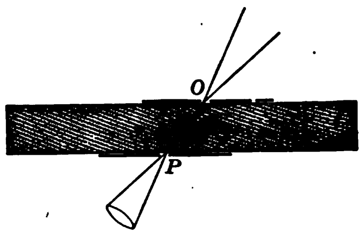


FIG. 484.

direction, forming a conical shell. To test this, a beam of light, converging at the proper angle, was allowed to fall on the piece of aragonite at *O* (Fig. 484), and emerge through a small aperture, *P*, in a piece of tin foil, which cut off all rays except those transmitted along *OP*. The relative positions of *O* and *P* were slowly changed, and when the adjustment was complete, a bright annulus of light was seen on looking into the aperture at *P*. This phenomenon is known as external conical refraction.

634. Interference of Polarized Light. — The experiments of Fresnel and Arago show that in order that two beams of polarized light shall interfere in the same manner as ordinary light, it is necessary: 1°, that the beams of light shall be polarized in the same plane; and, 2°, that they have a common origin.

If, when the analyzer is set so as to extinguish a beam, a thin section of a doubly refracting crystal is interposed between the analyzer and the polarizer, the light will be restored and also colored, the hue depending on the thickness of the crystal. By turning the plate in its own plane, two positions will be found in which the light is extinguished, and two for which it is a maximum. If, on the other hand, the analyzer be revolved, the saturation will diminish till white is reached, when the color changes to the complementary hue and the saturation increases. If the analyzer be replaced by a double image prism, the two fields will have complementary colors, except at the spot where they overlap, which will be white.

The explanation is as follows: Let *CP* (Fig. 485) represent the direction and amplitude of vibration of the incident plane polarized light; *SS*, the principal plane of the thin crystal section; and *AA*, the principal plane of a doubly refract-

ing crystal, used as an analyzer. On entering the crystal section the plane polarized light is divided into two trains, which traverse the plate with different velocities, the ordinary component having an amplitude and direction of vibration, CO, and the extraordinary an amplitude and direction, CE. After passing the analyzer, both these component vibrations are subdivided into pairs, Co, Ce, and Co', Ce', which have been differently retarded and are in condition to interfere. The intensity of the ordinary and extraordinary image will, accordingly, be different for different wave-lengths; and if the incident light is white, the images will appear colored, but of complementary hues, since their sum should give the original set of waves, provided none are lost by absorption

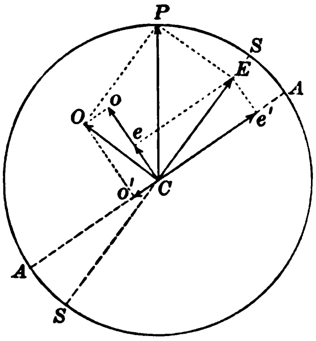


FIG. 485.

If the preceding experiment be varied by passing the light through two thin crystalline plates before it is received in the analyzer, the images will appear complementary as before, but the hues will change with the rotation of the analyzer, or of either plate.

635. Double Refraction in Strained Transparent Substances. — Any unequal alteration in the elasticity of a homogeneous transparent solid by a stress produces a double refraction, which is easily recognized when the body is examined by polarized light. Thus, for instance, if a block of glass, held by a clamp, as in Fig. 486, be placed before the


FIG. 486.

analyzer and examined by polarized light, it will be found that a slight pressure by the screw on the block brings out an elaborate colored pattern. Tempered glass, that is, glass which has been heated and suddenly cooled, likewise exhibits the presence of unequal stresses when examined by the polariscope.

In thick pieces of glass, such as are used for lenses of considerable size, it is difficult to anneal the blocks so that they will not show lack of uniformity by this sensitive test. Dr. Kerr has established the fact that fluid as well as solid dielectrics, when subjected to electrostatic stresses, are modified so as to become double refracting.

636. Rings and Cross. — If a plate cut from a uniaxial crystal be placed before the analyzer, and observed in convergent plane polarized light, a series of brilliantly colored rings interrupted by a rectangular bright or dark cross (Fig. 487) may be observed.

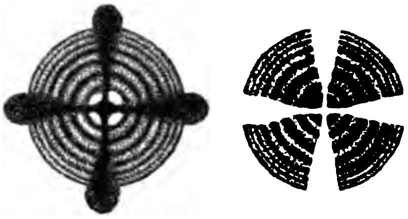


FIG. 487.

Let *MN* (Fig. 488) represent the crystal section, and suppose that light diverging from the point *O* falls on the plate in any direction, *OX*. Also, suppose that *PP* is the plane of vibration of the light transmitted by the polarizer, and *AA* that by the analyzer. When the light meets the surface at *X* it will be doubly refracted, the vibrations in one of the rays taking place at right angles to *OX*, since this line is in a principal plane of the crystal and the other in the plane of *OX*. When the point considered is at X' or X'', the crystal section evidently transmits only those vibrations which are parallel to *PP*, and these are completely extinguished by

the analyzer. Hence the regions PP and AA will appear dark. At any other points the crystal will exhibit the colors of thin plates, but arranged in rings about the axis, since the thickness of the plate traversed varies with the angle of in-

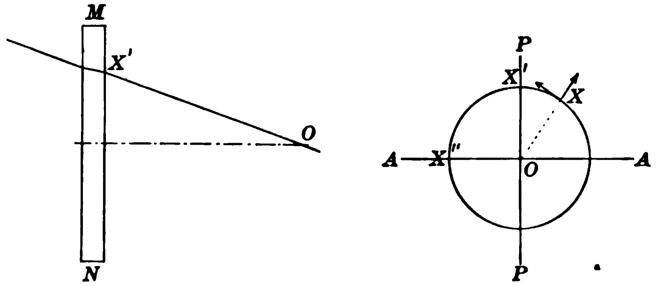


FIG. 488.

cidence. On rotating the analyzer 90° the cross appears white, and the colors change to their complementary value.

The analogous but more complicated phenomena presented by biaxial crystals are shown in Fig. 489. The succession

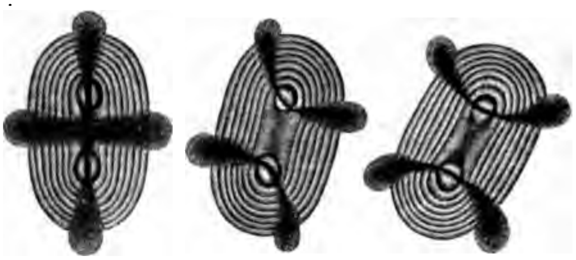


FIG. 489.

of forms illustrates the changes which occur as the crystal section is rotated 45° between the crossed polarizer and analyzer. The appearance of these figures is so varied and characteristic as to furnish, in practically all cases, a satisfactory means of identifying any doubly refracting crystal.

637. Circular Polarization.—Although, as has been pointed out, two beams of light polarized in planes at right angles cannot produce destructive interference, they may nevertheless combine so as to produce a vibration of a special character. The vibratory form resulting from such composition, when the periods are the same, has been shown in Art. 474 to be an ellipse inscribed in a rectangle whose sides are the component vibrations. Plane polarized light which has been passed through a doubly refracting plate and has acquired this special form of vibration is said to be elliptically polarized. Polarized light which has suffered reflection from a metal surface is also, in general, elliptically polarized.

When the difference of phase between the rectangular vibrations is just a quarter period and the amplitudes are the same, the vibrational form reduces to a circle. Circularly polarized light may be produced by passing plane polarized light through a thin sheet of mica, so chosen that the retardation of one train of waves over the other is just a quarter wave-length. Such a crystal section is known as a *quarter-wave plate.* The light which has passed through it will appear equally bright in all positions of the analyzer.

When a quarter-wave plate, used to produce circularly polarized light, is rotated 90° in its own plane, the light is still circularly polarized, but the sense of vibration in the circle will be reversed, *i.e.* right-hand circular polarization will be altered to left-hand, and the contrary.

638. Rotary Polarization.—Quartz is a uniaxial crystal in which the extraordinary wave surface lies completely within the ordinary wave surface, and the vibrations are, in general, elliptically polarized in opposite directions. In consequence, the speed of the two waves along the axis of the

crystal is different with opposite circular polarizations. It is easy to see that the resultant of two opposite circular vibrations of the same period is a simple harmonic vibration. Thus, if p, p' (Fig. 490) be two points moving with the same period in a circle, and symmetrically placed with respect to the line *MN*, then by resolving each circular motion into simple harmonic motions parallel and perpendicular to this line, it is seen that the perpendicular components annul each other, leaving a rectilinear vibration in the line *MN*. In an analogous manner the oppositely polarized beams through the quartz produce plane polarized light on emergence, but because one train of waves traveled faster than the other the plane of vibration of the emergent light will have been rotated through a certain angle proportional to the thickness of the quartz plate and depending on the wave-length of the light. If the incident light is white, the emergent light after passing the analyzer will exhibit colors depending on the position of its plane of vibration. Certain specimens of quartz and other crystals produce a rotation of the plane of polarization to the right and others to the left.

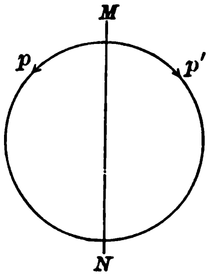


FIG. 490.

The rotation, for different wave-lengths, produced by a plate of quartz one millimeter thick, cut perpendicular to the axis, and having a temperature of 20° C., is shown in the following table.

	°		°
. . . .	12.67	E	27.54
B	15.75	F	32.77
C	17.32	G	42.60
D_1	21.68	H	51.19
D_2	21.73		

The rotations in yellow light for several other crystals are approximately as follows.

SUBSTANCE.	FORMULA.	α FOR 1mm THICKNESS.
		°
Cinnabar	HgS	32.5
Sodium Chlorate	$NaClO_3$	3.5
Sodium Bromate	$NaBrO_3$	2.8
Hyposulphate of Potash . .	$K_2S_2O_6$	8.4
" " Calcium .	$CaS_2O_6 + 4H_2O$	2.1
" " Lead . .	$PbS_2O_6 + 4H_2O$	5.5
" " Strontium	$SrS_2O_6 + 4H_2O$	1.6

639. Rotary Power of Liquids. — Many liquids and even vapors possess a power of rotating the plane of polarization similar to that of crystals, but in a far less degree.

In solids this power depends on the structure, and is lost when the body is fused or dissolved. In fluids, however, the power appears to be inherent in the molecule. The rotation produced by any active liquid increases directly as the thickness of the layer through which the light passes, and also varies with the wave-length of light used and with the temperature.

The rotary powers of liquids are usually stated in what is known as specific rotation, *i.e.* the rotation in degrees, per decimeter, per unit density. If α be the rotation of the plane of polarization in degrees, l the thickness of the layer of the fluid in decimeters, and ρ its density, the specific rotation $[\alpha]$ may be written

$$[\alpha] = \frac{\alpha}{l \cdot \rho}. \qquad (2)$$

When an active liquid is dissolved in an inactive one, the rotary power is also a function of the concentration of the

solution and of the nature of the solvent. If p denote the percentage composition of the solution, *i.e.* the number of grams of the solute in 100 grams of the solution, the specific rotation of a solution may be defined by

$$[\alpha] = \frac{100\alpha}{l \cdot p \cdot \rho} = \frac{100\alpha}{l \cdot c},$$

where c denotes the concentration or the number of grams of the solute in 100 cc. of the solution.

A knowledge of the rotary properties of substances furnishes a valuable method of investigating either the character or the concentration of a solution. This method of analysis is extensively used in the determination of sugars, whence it is commonly known as *saccharimetry.*

The specific rotations of a few sugars for sodium light are exhibited in the following table, rotation to the right being denoted by the plus, and rotation to the left by the minus sign.

NAME.	TEMP. C.°	LIMITS OF PERCENTAGE COMPOSITION, p.	$[\alpha]_D$.
Cane Sugar . . .	15	10 to 50	$+\ 66.94 - 0.012\,p$
Lactose	20	0 to 36	$+\ 52.53$ constant
Maltose	20	10	$+136.75$ to 136.96
Glucose	20	0 to 100	$+\ 47.73 + 0.0155\,p$
Levulose	20	2 to 30	$-\ 91.90 - 0.111\,p$

640. Saccharimetry. — An analyzer, such as a Nicol's prism, in which the position of the plane of polarization is judged by a maximum or minimum intensity of the field, is not sufficiently precise for quantitative measures in rotary polarization. For this end it is found that better results

may be obtained by modifying the apparatus so that the field appears double, with a different illumination in each portion, except at the critical position. The latter may be judged

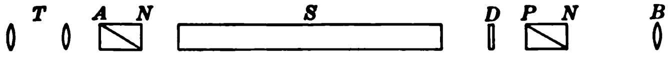


FIG. 491.

with considerable precision by the contrast in the two parts of the field.

Laurent's Saccharimeter. The arrangement of parts in Laurent's saccharimeter is shown in Fig. 491.

PN is a prism producing a beam of plane polarized light, and *S* a tube with glass ends, containing a solution whose rotary power is to be examined. *AN* is a Nicol analyzer,

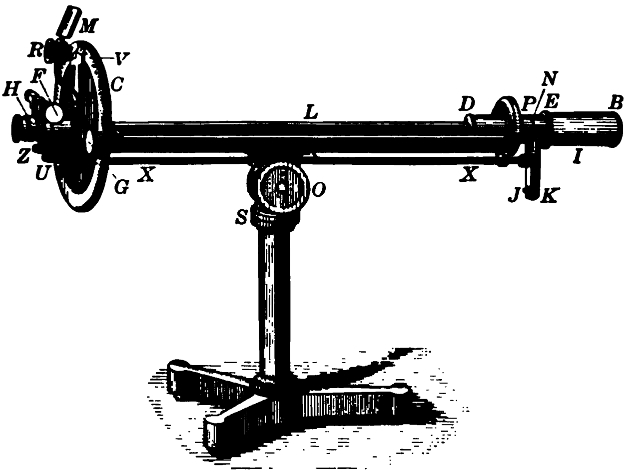


FIG. 492.

whose angular position is accurately indicated by the graduated circle and vernier, *V*, shown in Fig. 492, which represents the complete instrument. The distinguishing feature

of this apparatus is a disc, *D* (Fig. 491), one-half of which is a plane glass plate, *APB* (Fig. 493), and the other half a quartz plate, *AQB*, cut parallel to the axis and of such thickness that one train of waves is retarded a half wave-length over the other.

Suppose that *AB* is the direction of the optic axis of the quartz, and that *PO* is the plane of vibration of the incident light. This light will be transmitted by the glass plate without change, but it will be doubly refracted by the quartz, the component vibrations having directions respectively parallel and perpendicular to the axis. On emergence these components will re-combine to produce plane polarized light with its plane of vibration, *OQ*, rotated through an angle, 2*POB*, since the retardation of one component was just sufficient to reverse its direction.

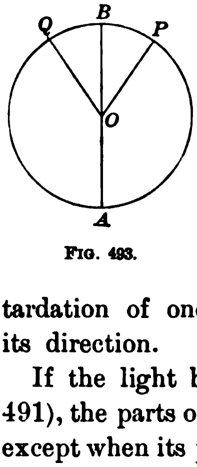


Fig. 493.

If the light be now examined by the analyzer *AN* (Fig. 491), the parts of the field will appear unequally illuminated, except when its principal plane bisects the angle between *OP* and *OQ*. The analyzer having been set so that both halves are equally bright, the substance to be examined is interposed at *S*. The angle through which the analyzer must be turned to produce equal illumination in both portions of the field is the angle through which the plane of polarization has been rotated. As this rotation would be different for different wave-lengths, the light used is either that emitted by a sodium-tinged flame, or the yellow obtained by sifting white light through a plate of bichromate of potash.

Soleil's Saccharimeter. Fig. 494 shows a type of saccharimeter invented by Soleil, which works upon a somewhat different principle from the preceding. The disc at *B* in

this case consists of two semicircular quartz plates of opposite sign and of same thickness. There will, in general, be some wave-length for which the rotation of the plane of polarization has been a right angle in each half of the field, so that its vibrations are now parallel to a certain line, say DD'.

White light, transmitted first through the Nicol P and then through the biquartz B, will thus, in general, appear

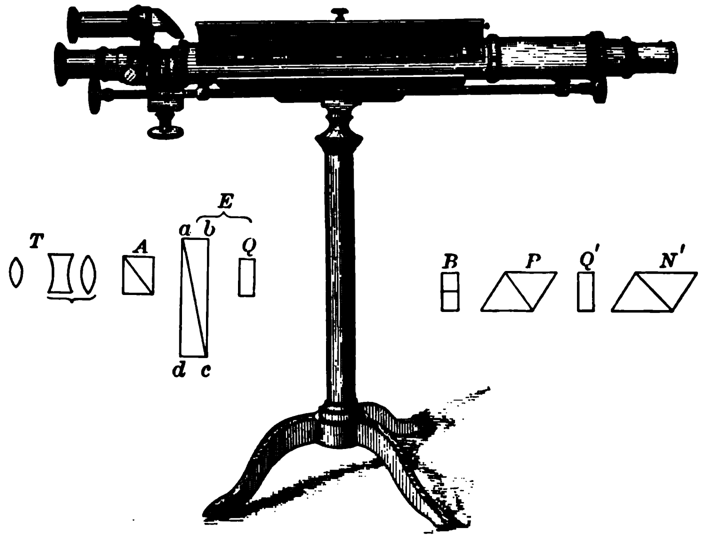


Fig. 494

colored, but of a different hue in each half of the field when examined through the analyzing Nicol A. If, however, the principal plane of the analyzer is perpendicular to DD' both parts of the field will have the same hue, because all the other wave-lengths are present in the same proportion in each half. By a proper choice of thickness of the quartz, this common color may be made any one desired. The hue usually selected is a violet, produced by the extinction of the

yellow waves in white light, and sometimes called the *sensitive tint*, since by a minute rotation of the analyzer in either direction one-half the field becomes blue and the other half red.

If the tube containing the solution to be examined be introduced between B and Q, when the field is of uniform hue, the consequent rotation of the plane of polarization will produce a marked disparity in the color of the two sections of the field. In order to determine the amount of this rotation, a *compensator* is provided at E, which consists of a right-handed quartz plate, Q, and two left-handed quartz wedges, *abc* and *adc*, arranged to slide one over the other, so as to vary their combined thickness by a measured amount. By a proper adjustment of the wedges it is clearly possible to produce such a right- or left-handed rotation of the plane of polarization as exactly to neutralize that produced by the solution. The value of a scale division on the compensator may be reduced to angular measure by an observation made on some substance having a known rotation.

The German instruments of this type are graduated so that one scale division corresponds to a rotation of the plane of polarization through 0.346°. For commercial tests, 26.0 gms. of cane sugar in 100 cc. of aqueous solution is termed a normal solution. It is assumed that if the sugar is pure a column of this solution 20 cm. long would produce a rotation, as read on the compensator scale, of 100 divisions. If the sugar is impure, the reading of the scale will give the percentage of saccharose present in the sugar.

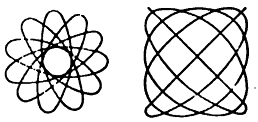
In order to adapt the instrument for use with a colored liquid, a *regulator*, consisting of a Nicol N' and a quartz plate, Q', is provided, so that the hue of the light entering the polarizer N may be altered at will. Thus, by selecting a hue complementary to that of the light transmitted by the substance, the so-called sensitive tint may always be obtained.

641. Magneto-Optic Rotation. — In 1845 Faraday discovered that a dense boro-silicate of lead, when placed in a strong magnetic field, acquired the ability to rotate the plane of polarization of light transmitted in the direction of the field. When the light was propagated in the same direction as the lines of force, the rotation was positive, as determined by the familiar right-handed screw rule; but when passed in the opposite direction, the rotation was negative. In a direction perpendicular to the field, the plane of polarization was unaltered.

Later experiments indicate that probably all substances are similarly modified by the magnetic forces.

It will be observed that there is this difference between magneto-optic rotation and that of crystals; namely, that if the light, having traversed a section of quartz in one direction, be reflected and made to traverse the plate in the opposite direction, the rotation will be undone, while in the case of the magnetic field the rotation is doubled.

642. Nature of Ordinary Light. — Common light does not exhibit any evidence of polarization. Accordingly, it must be assumed that the character of its vibrational form is continually changing, so that in a brief period these vibrations are performed in all azimuths — perhaps in a manner resembling *A* or *B* (Fig. 495). At least it is known that if the plane of polarization of light be revolved with a period less than the duration of visual impression, about a tenth of a second, all trace of polarization disappears. The phenomenon of interference bands, of which several thousands may be counted in homogeneous light, shows that the vibrations of light must possess a certain degree of uniformity;



A *B*

Fig. 495.

but since about fifty millions of vibrations are performed in a tenth of a second, it is obvious that there is room for great diversity which might entirely escape detection in the average impression made on the eye. Accordingly, if this irregularity be present in two streams of common light from different sources, or from different parts of the same source, there is no reason why the streams should interfere.

MISCELLANEOUS EXAMPLES.

1. A battery of 20 cells in series has been improperly set up. On introducing two similar cells, so that they reinforce or oppose the battery, the current in the circuit is found to be in the ratio of 4 to 3. How many cells of the battery were reversed? *Ans.* 3.

2. A piece of zinc weighing 48.3 gms., at a temperature of 10.7°, was immersed in a current of steam at 100° and found to condense 0.762 gm. of steam. What was the specific heat of the zinc?
Ans. 0.0947.

3. At what temperature will the pitch of a pipe, giving 124 vibrations per second at 10°, be raised a Fifth? *Ans.* 364° C.

4. A stone dropped from a cliff is heard to strike the ground 7.3 seconds later. Assuming the velocity of sound to be 1150 feet per sec., what is the hight of the cliff? *Ans.* 724 feet.

5. How many beats per second would be heard if $f'\sharp$ were simultaneously sounded on the natural and on the even tempered scale?
Ans. 1.6.

6. Notes of 225 and 336 vibrations per second, in which the first and second upper partials are present, are simultaneously sounded. What beats will be heard and whence do they arise?
Ans. 3 per sec.

7. In an experiment with Kundt's tube the length of the rubbed glass rod was 900 cm., and the nodal points were found at the marks 0, 63, 127, 188, 252, on a scale divided to centimeters. What was the velocity of sound in glass? *Ans.* $4880\ \frac{\text{met.}}{\text{sec.}}$.

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Hastings, C.S.
General physics.

NAME	DATE	NAME	DATE